MATH 135 – Calculus 1
L'Hopital's Rule
November 18, 2019

Background

Recall that we have defined the derivative of a function at x by using the limit:

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}.$$

One application of derivatives "turns this around" and *uses derivatives to compute limits of indeterminate forms* $0/0$ *or* ∞/∞ *or other limits that can be put into those forms.* The precise statement is a bit complicated, but this is really a reflection of the *power* of the result:

Theorem 1 (L'Hopital's Rule) *Let f and g be differentiable on an interval containing a and suppose that $f(a) = g(a) = 0$. Assume that $g'(x) \neq 0$ on an interval containing a, except possibly at $x = a$. Then*

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)} \tag{1}$$

if the limit on the right exists, or is equal to $\pm\infty$.

The same statement applies if $\lim_{x\to a} f(x) = \pm\infty$ and $\lim_{x\to a} g(x) = \pm\infty$. The statement also applies for one-sided limits. The idea that makes this work is that for x close to a (which is what matters in taking the limit) $y = f(x)$ is close to the tangent line $y = f(a)+f'(a)(x-a) = f'(a)(x-a)$ and $y = g(x)$ is close to the tangent line $y = g(a) + g'(a)(x-a) = g'(a)(x-a)$. So

$$\frac{f(x)}{g(x)} \doteq \frac{f'(a)(x-a)}{g'(a)(x-a)} = \frac{f'(a)}{g'(a)}.$$

(The first approximate equality is not an exact equality, of course. But the difference between $f(x)$ and $f'(a)(x-a)$ goes to zero "really fast" as $x \to a$ and $\frac{f(x)}{g(x)}$ goes to $\frac{f'(a)}{g'(a)}$ in the limit as $x \to a$.)

Important Note: Note that the rightside of 1 is the quotient of the derivatives. This is *different* from the derivative of the quotient $f(x)/g(x)$. We would use the quotient rule for that, and the result is not just $f'(x)/g'(x)$.

Some examples

1. Consider $\lim_{x\to 0} \frac{e^x-1}{\sin(x)}$. This is a $0/0$ indeterminate form since $\lim_{x\to 0} e^x - 1 = 0 = \lim_{x\to 0} \sin(x)$. So we can apply L'Hopital:

$$\lim_{x\to 0} \frac{e^x - 1}{\sin(x)} = \lim_{x\to 0} \frac{e^x}{\cos(x)} = 1.$$

2. Consider $\lim_{x\to\infty} \frac{\ln(x)}{x^{1/2}}$. This is an ∞/∞ indeterminate form. L'Hopital's Rule also applies to this sort of limit, and we get

$$\begin{aligned} \lim_{x\to\infty} \frac{\ln(x)}{x^{1/2}} &= \lim_{x\to\infty} \frac{\frac{1}{x}}{\frac{1}{2x^{1/2}}} \\ &= \lim_{x\to\infty} \frac{2}{x^{1/2}} \\ &= 0. \end{aligned}$$

3. (Sometimes we might need to apply L'Hopital more than once. That is OK!) For instance, consider

$$\lim_{x\to 0} \frac{\sin(x) - x}{x^3},$$

which is 0/0 indeterminate. Applying L'Hopital twice, we get:

$$\begin{aligned}
\lim_{x\to 0} \frac{\sin(x) - x}{x^3} &= \lim_{x\to 0} \frac{\cos(x) - 1}{3x^2} \quad \text{still } 0/0! \\
&= \lim_{x\to 0} \frac{-\sin(x)}{6x} \\
&= \frac{-1}{6}.
\end{aligned}$$

The last equation comes because of the trig limit we discussed earlier: $\lim_{x\to 0} \frac{\sin(x)}{x} = 1$.

4. Some other indeterminate forms such as 0^0 or 1^∞ can be evaluated by taking logarithms first, using L'Hopital, then exponentiating the result. For instance, consider

$$\lim_{x\to 0} (1 + x)^{1/x},$$

a 1^∞ form. Taking logs to start, we have

$$\begin{aligned}
\lim_{x\to 0} \ln\left((1 + x)^{1/x}\right) &= \lim_{x\to 0} \frac{\ln(1 + x)}{x} \quad \text{(a } 0/0 \text{ form)} \\
&= \lim_{x\to 0} \frac{1}{x + 1} \\
&= 1.
\end{aligned}$$

Since we took the natural logarithm to start, we need to exponentiate to get the final result:

$$\lim_{x\to 0} (1 + x)^{1/x} = e^1 = e.$$

Note: Some calculus books even take this limit as the definition of the number e.

Practice Problems

1. $\lim_{x\to\infty} \frac{x}{e^x}$

Solution: This is an ∞/∞ form, so L'Hopital's Rule applies. Applying the rule,

$$\begin{aligned}
\lim_{x\to\infty} \frac{x}{e^x} &= \lim_{x\to\infty} \frac{1}{e^x} \\
&= 0.
\end{aligned}$$

2. $\lim_{x\to 2} \frac{x^3 - 5x^2 + 7x - 4}{x^3 - x^2 - 8x + 12}$

Solution: This is not a 0/0 form, so L'Hopital's Rule *does not apply* The limit DNE because $\lim_{x\to 2} x^3 - 5x^2 + 7x - 4 = 8 - 20 + 14 - 4 = -2$, but $\lim_{x\to 2} x^3 - x^2 - 8x + 12 = 8 - 4 - 16 + 12 = 0$.

3. $\lim_{x\to 0} \frac{\tan(x)\sin(x)}{x^2}$

Solution: This is a 0/0 form, so L'Hopital's Rule does apply. After the first application, the limit is still 0/0, so we "do it again."

$$\begin{aligned}
\lim_{x\to 0} \frac{\tan(x)\sin(x)}{x^2} &= \lim_{x\to 0} \frac{\sec^2(x)\sin(x) + \tan(x)\cos(x)}{2x} \\
&= \lim_{x\to 0} \frac{\sec^2(x)\cos(x) + 2\sin(x)\sec^2(x)\tan(x) - \tan(x)\sin(x) + \sec^2(x)\cos(x)}{2} \\
&= \lim_{x\to 0} \frac{1\cdot 1 + 0 - 0 + 1}{2} \\
&= 1
\end{aligned}$$

4. $\lim_{x\to 0} (\cos(x))^{3/x^2}$ (Hint: Take logarithms first.)

Solution: This is a 1^∞ indeterminate form. Taking the hint, then applying L'Hopital to the 0/0 limit we get by doing that:

$$\begin{aligned}
\lim_{x\to 0} \ln\left((\cos(x))^{3/x^2}\right) &= \lim_{x\to 0} \frac{3\ln(\cos(x))}{x^2} \\
&= \lim_{x\to 0} \frac{-3\sin(x)}{2x\cos(x)} \quad \text{still } 0/0 \\
&= \lim_{x\to 0} \frac{-3\cos(x)}{2\cos(x) - 2x\sin(x)} \\
&= -\frac{3}{2}.
\end{aligned}$$

The limit is then $e^{-3/2}$.