MATH 135 – Calculus 1
The MVT and Consequences
November 13, 2019

Background

Recall from today's video, recall that we now know a statement called the Mean Value Theorem (MVT):

Theorem 1 *(MVT) Let $f(x)$ be continuous on $[a,b]$ and differentiable on (a,b). Then there exists $c \in (a,b)$ such that*

$$f'(c) = \frac{f(b)-f(a)}{b-a}.$$

The MVT lets us show in a "rigorous" (i.e. complete and convincing) way that some of the patterns we have discussed intuitively actually always hold. For instance

- If $f'(x) > 0$ on an interval (a,b), then f is increasing on that interval.
- If $f'(x) < 0$ on an interval (a,b), then f is decreasing on that interval.
- If $f'(x) = 0$ on an interval (a,b), then f is constant on that interval.

In addition, we get the following way to determine whether a critical point c of f is a local maximum or local minimum:

Theorem 2 *(First Derivative Test) Let c be a critical point of a function f.*

- *If f' changes sign from positive to negative at c, then $f(c)$ is a local maximum of f.*
- *If f' changes sign from negative to positive at c, then $f(c)$ is a local minimum of f.*
- *If f' does not change sign at c, then $f(c)$ is neither a local maximum nor a local minimum.*

Questions

1. Show that the conclusion of the Mean Value Theorem is true for $f(x) = x^3 - 12x^2 + 21x$ on the interval $[-1, 10]$. That is, find all the $c \in (-1, 10)$ for which $f'(c) = \frac{f(10)-f(-1)}{10-(-1)}$. There is a plot of the graph $y = f(x)$ on the back of this page. Draw in lines illustrating the conclusion of the Mean Value Theorem.

   Solution: We have $f'(x) = 3x^2 - 24x + 21$. Since $f(10) = 10$ and $f(-1) = -34$, the slope of the secant line through the endpoints is

   $$\frac{f(10)-f(-1)}{10-(-1)} = \frac{10+34}{11} = \frac{44}{11} = 4$$

   Setting $f'(x) = 4$ we get the quadratic equation $3x^2 - 24x + 17 = 0$. We need the quadratic equation for this:

   $$x = \frac{24 \pm \sqrt{(-24)^2 - 4 \cdot 3 \cdot 17}}{2} = 4 \pm \frac{\sqrt{93}}{3} \doteq .7854, 7.2146$$

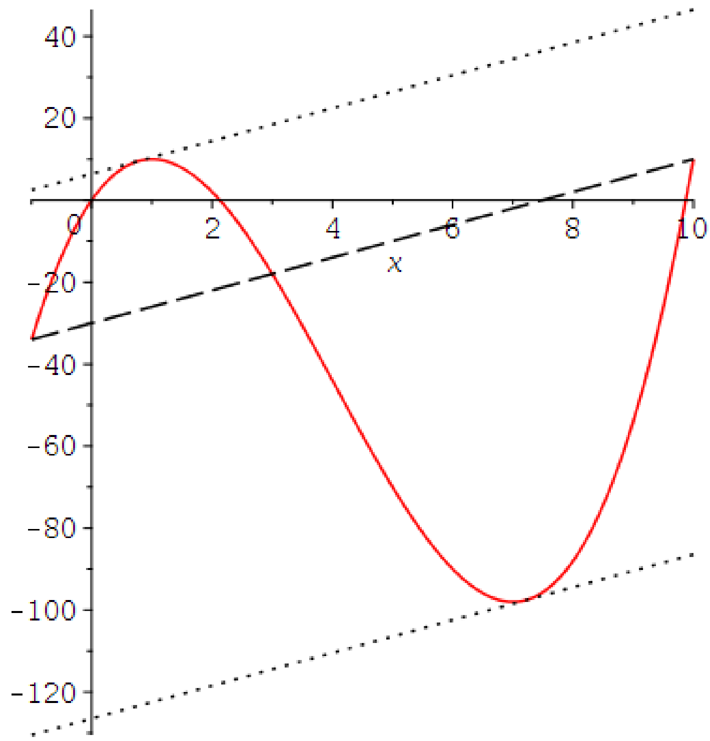

Figure 1: Plot for questions 1 and 3

The tangent lines to the graph $y = f(x)$ at these two x-values are *parallel* to the secant line through $(-1, f(-1)) = (-1, -34)$ and $(10, f(10)) = (10, 10)$. See plot on back of the page.

2. Slim Shady drove through the EZPass lane and got on the Mass Pike at Allston at 11:00am one morning. He drove west to Auburn and exited through the EZPass lane there at 11:37am. The distance between the two exits is 42 miles. Two weeks later, Slim received a speeding ticket from the Mass State Police for $150 in the mail. Should he try to fight this in court, or is the ticket justified? Explain. (Note: The maximum posted speed is 65 miles per hour the whole way.)

   Solution: His average velocity while on the Mass Pike was

   $$\frac{42}{37} \text{ miles per minute} = \frac{42}{37} \times 60 \text{ (miles per minute)(minutes per hour)} = 68.1 \text{ miles per hour.}$$

   That means that he had to be going over the speed limit for a significant portion of the trip. Busted!

3. Now suppose that the graph on the back is $y = g'(x)$ for some other function $g(x)$. What are the critical points of $g(x)$? Classify each of them as a local maximum or local minimum using the First Derivative Test above.

   Solution: The critical points are $x = 0$ and $x = 2.1$, $x = 9.9$ (approximately). By the First Derivative Test, g has local minima at $x = 0$ and $x = 9.9$ since the graph of g' is crossing the x-axis from negative to positive there; g has a local maximum at $x = 2.1$ since the graph of g' is crossing the x-axis from positive to negative there.