MATH 135 – Calculus 1
Review for Final
December 13, 2019

Practice Questions

1. What is the limit definition of the derivative $f'(x)$?

   Solution:
   $$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h},$$
   provided that the limit exists.

2. Use the definition to compute $f'(x)$ for $f(x) = \sqrt{x-4}$.

   Solution:
   $$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{\sqrt{x+h-4}-\sqrt{x-4}}{h} \\ &= \lim_{h\to 0} \frac{\sqrt{x+h-4}-\sqrt{x-4}}{h} \cdot \frac{\sqrt{x+h-4}+\sqrt{x-4}}{\sqrt{x+h-4}+\sqrt{x-4}} \\ &= \lim_{h\to 0} \frac{(x+h-4)-(x-4)}{h(\sqrt{x+h-4}+\sqrt{x-4})} \\ &= \lim_{h\to 0} \frac{h}{h(\sqrt{x+h-4}+\sqrt{x-4})} \\ &= \lim_{h\to 0} \frac{1}{\sqrt{x+h-4}+\sqrt{x-4}} \\ &= \frac{1}{2\sqrt{x-4}}. \end{aligned}$$

   (Note that this agrees with the formula you would get by applying the chain rule to $f(x) = (x-4)^{1/2}$.)

3. What is the equation of the tangent line to $y = \sqrt{x-4}$ at $x = 8$?

   The equation is
   $$y - 2 = \frac{1}{2\sqrt{4}}(x-8)$$
   or
   $$y = \frac{1}{4}x.$$

4. Suppose $f(x)$ is a function such that $f'(x)$ and $f''(x)$ exist and are continuous for all real x. Assume that $f'(x) > 0$ on $(-3,-2)$ and $(2,\infty)$, while $f'(x) < 0$ on $(-\infty,-3)$ and $(-2,2)$. Also assume $f'(-3) = f'(-2) = f'(2) = 0$.

   (a) What does the First Derivative Test tell you about $x = -3$, $x = -2$, and $x = 2$?

   Solution: Since f' changes from negative to positive at $x = -3$, f has a local minimum there. Similarly, f has local maximum at $x = -2$, and another local minimum at $x = 2$

(b) Using the Mean Value Theorem, explain why f must have a point of inflection on the interval $(-3,-2)$ and another point of inflection on the interval $(-2,2)$.

Solution: Recall that the Mean Value Theorem says that if g is continuous on $[a,b]$ and differentiable on (a,b), then there exists a c in (a,b) where

$$g'(c) = \frac{g(b)-g(a)}{b-a}.$$

If we apply this to $g = f'$, then since $g(-3) = g(-2) = 0$, there must be some point c in $(-3,-2)$ where $g'(c) = f''(c) = 0$. The same reasoning applies in the interval $(-2,2)$ as well.

(c) Suppose you also know $f(-3) = 4$, $f(-2) = 6$, $f(2) = 0$, and

$$\lim_{x\to\pm\infty} f(x) = 12.$$

Sketch a possible graph $y = f(x)$ that satisfies all of these conditions.

Solution: Omitted. There are infinitely many correct graphs that satisfy all of these conditions.

5. Let $f(x) = \frac{20x}{x^2-4x+3}$, for which

$$\begin{aligned} f'(x) &= \frac{-20x^2+60}{(x^2-4x+3)^2} \\ f''(x) &= \frac{40x^3-360x+480}{(x^2-4x+3)^3} \end{aligned}$$

(a) Where does $y = f(x)$ have vertical asymptotes? Does it have a horizontal asymptote? If so, where?

Solution Since $x^2-4x+3 = (x-1)(x-3)$ there are vertical asymptotes at $x = 1$ and $x = 3$. We must understand $\lim_{x\to\pm\infty} f(x)$ to determine horizontal asymptotes. By L'Hopital's Rule,

$$\lim x \to \pm\infty \frac{x}{x^2-4x+3} = \lim_{x\to\pm\infty} \frac{1}{2x-4} = 0$$

There is a horizontal asymptote at the line $y = 0$.

(b) What are

$$\lim_{x\to 1^-} f(x) \text{ and } \lim_{x\to 1^+} f(x)?$$

What about

$$\lim_{x\to 3^-} f(x) \text{ and } \lim_{x\to 3^+} f(x)?$$

Solution: By comparing the signs of $20x$ and x^2-4x+3 on either side of $x = 1$ and $x = 3$, we can see

$$\lim_{x\to 1^-} f(x) = +\infty \text{ and } \lim_{x\to 1^+} f(x) = -\infty$$

Similary

$$\lim_{x\to 3^-} f(x) = -\infty \text{ and } \lim_{x\to 3^+} f(x) = +\infty?$$

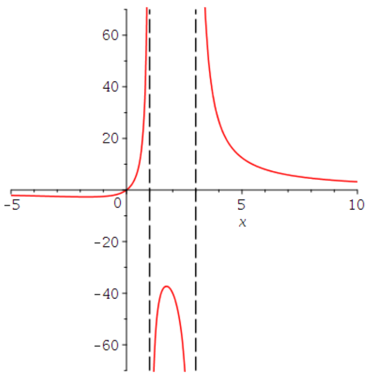

Figure 1: $y = \frac{20x}{x^2-4x+3}$ for $-5 \le x \le 10$

(c) Does $f(x)$ have any critical points? Where are they located? What are the critical values?

Solution: There are critical points at the solutions of $f'(x) = 0$ – namely where $-20x^2 + 60 = 0$, so $x = \pm\sqrt{3}$. The critical values are $f(\sqrt{3}) \doteq -37.3$ and $f(-\sqrt{3}) \doteq -2.68$.

(d) What is the concavity of the graph $y = f(x)$ on the interval $(3, \infty)$?

Solution: At all x in that interval, for instance at $x = 4$, $f''(x) > 0$. So the graph $y = f(x)$ is concave up.

(e) Sketch the graph $y = f(x)$.

Solution: See Figure 1 above.