MATH 135 – Calculus 1
Solutions/Answers for Exam 3 Practice Problems
November 18, 2016

I. Find the indicated derivative(s) and simplify.

(A)

$$y = \ln(x)\left(x^7 - \frac{4}{\sqrt{x}}\right)$$

Solution: By the product rule and the derivative rules for $\ln(x)$ and powers:

$$y' = \ln(x)(7x^6 + 2x^{-3/2}) + x^6 - \frac{4}{x^{3/2}}$$

(B)

$$y = \sin^{-1}(e^{2x} + 2)$$

Solution: By the chain rule and the inverse sine derivative rule:

$$y' = \frac{1}{\sqrt{1-(e^{2x}+2)^2}} \cdot 2e^{2x} = \frac{2e^{2x}}{\sqrt{1-(e^{2x}+2)^2}}$$

(C)

$$y = \frac{\ln(x+1)}{3x^4 - 1}$$

Solution: Using the quotient rule:

$$y' = \frac{(3x^4-1)\cdot\frac{1}{x+1} - \ln(x+1)(12x^3)}{(3x^4-1)^2}$$

(D)

$$y = \frac{\sin(x)}{1+\cos(x)}$$

Solution: By the quotient rule:

$$y' = \frac{(1+\cos(x))\cos(x) + \sin^2(x)}{(1+\cos(x))^2} = \frac{1}{1+\cos(x)}$$

(E)

$$y = \tan^{-1}(x^2 + x)$$

Solution: By the inverse tangent rule:

$$\frac{dy}{dx} = \frac{2x+1}{1+(x^2+2x)^2}$$

(F) Using implicit differentiation:

$$xy^2 - 3y^3 + 2x^4 - 4xy = 2$$

Solution: We have

$$2xy\frac{dy}{dx} + y^2 - 9y^2\frac{dy}{dx} + 8x^3 - 4x\frac{dy}{dx} - 4y = 0$$

so

$$\frac{dy}{dx} = \frac{4y - 8x^3 - y^2}{2xy - 9y^2 - 4x}$$

(G) Find the equation of the line tangent to the curve from (F) at $(x, y) = (1, 0)$

Solution: The slope is 2, so $y = 2x - 2$.

(H) Find

$$\frac{d}{dx}\left(5x\sqrt{x} - \frac{2}{x^3} + 11x - 4\right)$$

Solution: The function can also be written as $5x^{3/2} - 2x^{-3} + 11x - 3$. In this form, we only need the power rule to differentiate:

$$y' = \frac{15}{2}x^{1/2} + 6x^{-4} + 11.$$

(I)

$$\frac{d}{dt}\left(\frac{t^2e^{3t}}{t^4+1}\right)$$

Solution: By the quotient rule, product rule, and chain rule the derivative is:

$$\frac{(t^4+1)\cdot(3t^2e^{3t} + 2te^{3t}) - (t^2e^{3t})\cdot(4t^3)}{(t^4+1)^2} = \frac{e^{3t}(3t^6 - 2t^5 + 3t^2 + 2t)}{(t^4+1)^2}.$$

(J)

$$\frac{d^2}{dz^2}\frac{z^2 - 2z + 4}{z^2 + 1}$$

Solution: Again using the quotient rule, the first derivative is:

$$\frac{(z^2+1)(2z-2) - (z^2 - 2z + 4)(2z)}{(z^2+1)^2} = \frac{2z^2 - 6z - 2}{(z^2+1)^2}.$$

So then differentiating again, the second derivative is

$$\begin{aligned}
&= \frac{(z^2+1)^2(4z-6) - (2z^2 - 6z - 2)(2)(z^2+1)(2z)}{(z^2+1)^4} \\
&= \frac{(z^2+1)(4z-6) - (4z)(2z^2 - 6z - 2)}{(z^2+1)^3} \\
&= \frac{-4z^3 + 18z^2 + 12z - 6}{(z^2+1)^3}.
\end{aligned}$$

(K)

$$\frac{d}{dx}\left(\sin(x)\left(x^7 - \frac{4}{\sqrt{x}}\right)\right)$$

Solution: Rewrite the fuction as $\sin(x)(x^7 - 4x^{-1/2})$. Then by the product rule the derivative is:

$$\sin(x)(7x^6 + 2x^{-3/2}) + (x^7 - 4x^{-1/2})\cos(x).$$

(L) Find y' (note this is just another way of asking the same question!)

$$y = (e^{2x} + 2)^3$$

By the chain rule, the derivative is:

$$3(e^{2x} + 2)^2(2e^{2x}) = 6e^{2x}(e^{2x} + 2)^3.$$

(M) Find y' and y''

$$y = \frac{x+1}{3x^4 - 1}$$

Solution: By the quotient rule,

$$y' = \frac{(3x^4 - 1)(1) - (x+1)(12x^3)}{(3x^4 - 1)^2} = \frac{-9x^4 - 12x^3 - 1}{(3x^4 - 1)^2}.$$

So then differentiating again with the quotient rule, we get

$$y'' = \frac{(12x^2)(9x^5 + 15x^4 + 5x + 3)}{(3x^4 - 1)^3}$$

(There is a common factor of $3x^4 - 1$ that can be cancelled between the numerator and the denominator after you apply the quotient rule the second time.)

(N) Find y'

$$y = \frac{\sin(x)}{1 + \cos(x)} + x^2\cos(x^3 + 3)$$

Solution: By the quotient, product, and chain rules:

$$\begin{aligned} y' &= \frac{(1+\cos(x))\cos(x) - \sin(x)(-\sin(x))}{(1+\cos(x))^2} - x^2\sin(x^3+3)(3x^2) + 2x\cos(x^3+3) \\ &= \frac{1+\cos(x)}{(1+\cos(x))^2} - 3x^4\sin(x^3+3) + 2x\cos(x^3+3) \\ &= \frac{1}{1+\cos(x)} - 3x^4\sin(x^3+3) + 2x\cos(x^3+3). \end{aligned}$$

II. The total cost (in \$) of repaying a car loan at interest rate of $r\%$ per year is $C = f(r)$.

(A) What is the meaning of the statement $f(7) = 20000$?

Solution: At an interest rate of 7% per year, the cost of repaying the loan is 20000 dollars.

(B) What is the meaning of the statement $f'(7) = 3000$? What are the units of $f'(7)$?

Solution: At an interest rate of 7% per year, the rate of change of the cost of repaying the loan is 3000 dollars per (% per year).

III. The quantity of a reagent present in a chemical reaction is given by $Q(t) = t^3 - 3t^2 + t + 30$ grams at time t seconds for all $t \geq 0$. (Note: For a question like this, I could also give you the plot of the function and ask questions like those below. In this case you need to start from the formula and compute $Q'(t)$; if you were given the graph, you need to make the connection between slopes of tangent lines and signs of $Q'(t)$ visually.)

(A) Over which intervals with $t \geq 0$ is the amount increasing? (i.e. $Q'(t) > 0$) decreasing (i.e. $Q'(t) < 0$)?

Solution: $Q'(t) = 3t^2 - 6t + 1$. $Q'(t) = 0$ when

$$t = \frac{6 \pm \sqrt{36 - 12}}{6} = 1 \pm \frac{\sqrt{6}}{3} \doteq 1.816, .184.$$

Since this is a quadratic function with a positive t^2 coefficient, $Q'(t) > 0$ for $t > 1.816$ and $t < .184$. $Q'(t) < 0$ for $.184 < t < 1.816$ (t in seconds).

(B) Over which intervals is the rate of change of Q increasing? decreasing?

Solution: The rate of change of Q is increasing when $(Q')' > 0$ and decreasing when $(Q')' < 0$. The second derivative of Q is $Q''(t) = 6t - 6$. So $Q''(t) > 0$ for $t > 1$ and $Q''(t) < 0$ for $t < 1$ (t in seconds).

IV. Compute the following limits using L'Hopital's Rule or other methods, as appropriate.

(A) $\lim_{x \to 0^+} x^{1/3} \ln(x)$

Solution: This is a $0 \cdot \infty$ form. We can write it as

$$\lim_{x \to 0^+} \frac{\ln(x)}{x^{-1/3}}$$

which makes it an ∞/∞ form. Applying L'Hopital's Rule, this is

$$\begin{aligned} \lim_{x \to 0^+} \frac{1/x}{(-1/3)x^{-4/3}} &= \lim_{x \to 0^+} -3 \cdot \frac{x^{4/3}}{x} \\ &= \lim_{x \to 0^+} -3x^{1/3} = 0. \end{aligned}$$

(B) $\lim_{x \to \infty} \frac{x^3}{e^{2x}}$

Solution: This is an ∞/∞ form. Applying L'Hopital's Rule three times:

$$\begin{aligned} \lim_{x \to \infty} \frac{x^3}{e^{2x}} &= \lim_{x \to \infty} \frac{3x^2}{2e^{2x}} \quad \text{still } \infty/\infty \\ &= \lim_{x \to \infty} \frac{6x}{4e^{2x}} \quad \text{still } \infty/\infty \\ &= \lim_{x \to \infty} \frac{6}{8e^{2x}} \quad \text{not } \infty/\infty \\ &= 0 \end{aligned}$$

(C) $\lim_{x\to 1} \dfrac{x^2-5x+4}{x^2-3x+2}$

Solution: This one is 0/0. We can either use L'Hopital:

$$\lim_{x\to 1} \frac{x^2-5x+4}{x^2-3x+2} = \lim_{x\to 1} \frac{2x-5}{2x-3} = \frac{-3}{-1} = 3.$$

Or we can also factor the top and bottom and cancel:

$$\lim_{x\to 1} \frac{x^2-5x+4}{x^2-3x+2} = \lim_{x\to 1} \frac{(x-1)(x-4)}{(x-1)(x-2)} = \lim_{x\to 1} \frac{x-4}{x-2} = \frac{-3}{-1} = 3.$$

(D) $\lim_{x\to\infty} \left(1+\dfrac{4}{x}\right)^x$

Solution: This is a 1^∞ form. We take logarithms, then apply L'Hopital's Rule:

$$\begin{aligned}
\lim_{x\to\infty} \ln\left(\left(1+\frac{4}{x}\right)^x\right) &= \lim_{x\to\infty} x\ln\left(1+\frac{4}{x}\right) \quad \text{this is } \infty\cdot 0 \\
&= \lim_{x\to\infty} \frac{\ln\left(1+\frac{4}{x}\right)}{\frac{1}{x}} \quad \text{(this is 0/0)} \\
&= \lim_{x\to\infty} \frac{\frac{1}{1+\frac{4}{x}}\cdot\frac{-4}{x^2}}{\frac{-1}{x^2}} \\
&= \lim_{x\to\infty} \frac{4}{1+\frac{x}{4}} \\
&= 4
\end{aligned}$$

Since we took logarithms to get the function whose limit is 4, the original limit is then found by exponentiating:

$$\lim_{x\to\infty} \left(1+\frac{4}{x}\right)^x = e^4.$$

V. All parts of this question refer to $f(x) = 4x^3 - x^4$.

(A) Find and classify all the critical points of f using the First Derivative Test.

Solution: $f'(x) = 12x^2 - 4x^3 = 4x^2(3-x)$. This is defined for all x and equal to zero at $x=0$ and $x=3$. Note that $4x^2 \geq 0$ for all x. So the sign of $f'(x)$ comes from the $3-x$ factor. That is negative for $x>3$ and positive for $x<3$. Hence f' changes sign from positive to negative at $x=3$ and the First Derivative Test says f has a local local maximum at $x=3$. On the other hand, $f'(x)$ does not change sign at $x=0$, so that critical point is neither a local maximum nor a local minimum.

(B) Over which intervals is the graph $y=f(x)$ concave up? concave down?

Solution: $f''(x) = 24x - 12x^2 = 12x(2-x)$, which is zero at $x=0$ and $x=2$. Then $f''(x) > 0$ and the graph $y=f(x)$ is concave up on $(0,2)$ and $f''(x) < 0$ and the graph $y=f(x)$ is concave down on $(-\infty,0)$ and $(2,\infty)$.

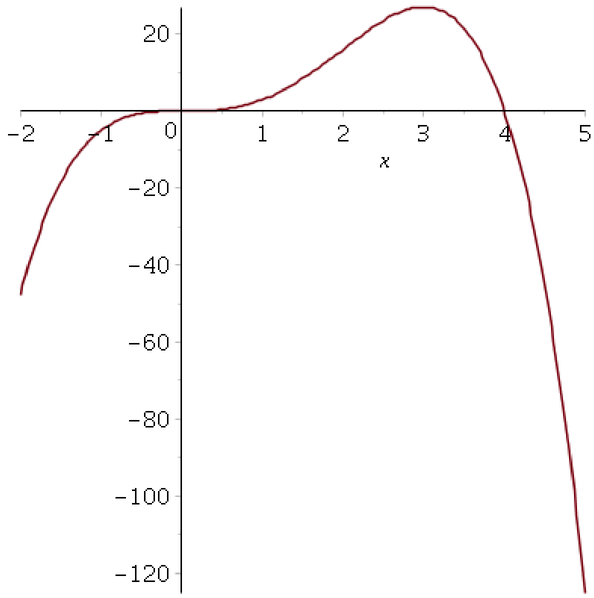

Figure 1: Plot of $y = f(x)$ for Problem V

(C) Sketch the graph $y = f(x)$.

Solution: See Figure 1 on the back of this page.

(D) Find the absolute maximum and minimum of $f(x)$ on the interval $[1, 4]$.

Solution: Only the critical point $x = 3$ is in this interval. $f(1) = 8$, $f(3) = 27$ and $f(4) = 0$. So $f(3) = 27$ is the maximum value and $f(4) = 0$ is the minimum value on the interval $[1, 4]$.

VI. All three parts of this question refer to the function $f(x)$ *whose derivative is plotted in Figure 1. NOTE: This is the graph* $y = f'(x)$ *not* $y = f(x)$.

(A) Give approximate values for all the critical points of $f(x)$ in the interval shown, and say whether f has a local maximum, a local minimum, or neither at each.

Solution: By inspection of the plot in Figure 1, we see that $f'(x) = 0$ at approximately $x = -5.2, -2$, and 1.2. Since f' changes sign from $+$ to $-$ at $x = -5.2$ and $x = 1.2$, those are local maxima for f. Since f' changes sign from negative to positive at $x = -2$, that is a local minimum for f.

(B) Find approximate values for all the inflection points of $f(x)$.

Solution: $y = f(x)$ has inflection points where f' changes from increasing to decreasing. That happens here at roughly $x = -3.9$ and $x = -0.8$.

(C) Over which intervals is $y = f(x)$ concave up? concave down?

Solution: Following on from (B), $y = f(x)$ will be concave down on intervals where $f'(x)$ is decreasing – roughly $(-6, -3.9)$ and $(-0.8, 2)$. $y = f(x)$ will be concave up on intervals where $f'(x)$ is increasing – roughly $(-3.9, -0.8)$.

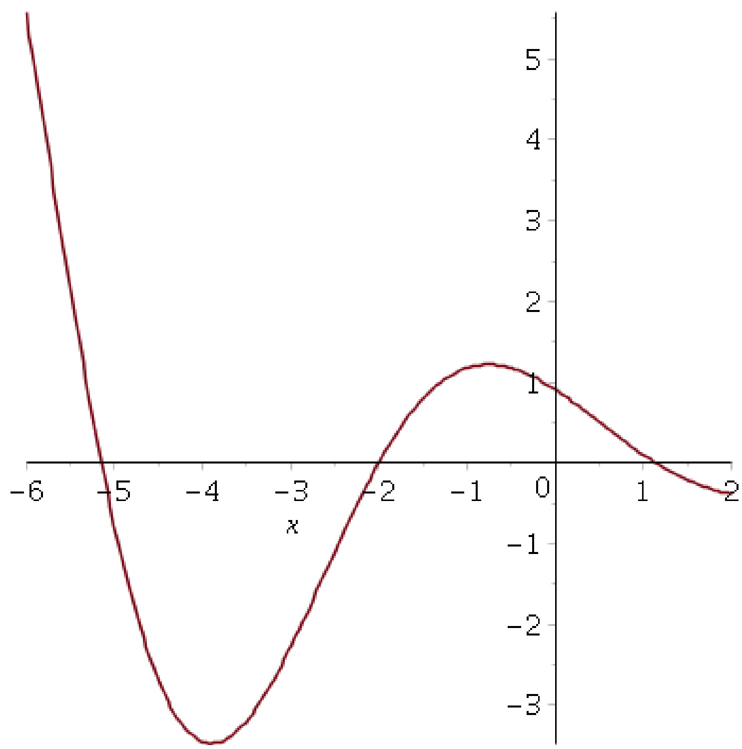

Figure 2: Plot of $y = f'(x)$ for Problem VI

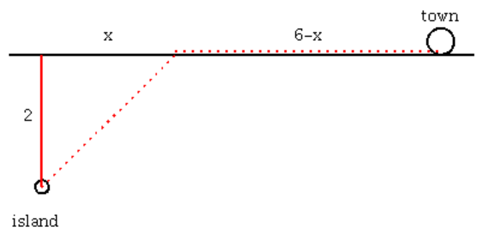

Figure 3: Figure for problem VII.

VII. A town wants to build a pipeline from a water station on a small island 2 miles off the shore of its water reservoir to the town. One possible route is shown dotted in red. The town is 6 miles along the shore from the point nearest the island. It costs \$3 million per mile to lay pipe under the water and \$2 million per mile to lay pipe along the shoreline.

A) Give the cost $C(x)$ of constructing the pipeline as a function of x.

Solution: By the Pythagorean theorem and the given information about cost per mile, we have

$$C(x) = 3\sqrt{4+x^2} + 2(6-x)$$

(a) B) Where along the shoreline should the pipeline hit land to minimize the costs of construction?

Solution: To find the minimum of $C(x)$, we can restrict to x in the closed interval $[0,6]$, since it clearly does no good to take $x < 0$ or $x > 6$. The function $C(x)$ has a critical number for $x > 0$ at the positive solution of $C'(x) = 0$:

$$\begin{aligned} 0 &= \frac{3x}{\sqrt{4+x^2}} - 2, \text{or} \\ 3x &= 2\sqrt{4+x^2} \\ 9x^2 &= 16 + 4x^2 \\ 5x^2 &= 16 \\ x &= \frac{4}{\sqrt{5}} \doteq 1.79. \end{aligned}$$

We have $C(0) = 18$, $C(6) = 3\sqrt{40} \doteq 19.0$, and $C\left(\frac{4}{\sqrt{5}}\right) \doteq 16.47$. So the minimum cost is attained at $x = \frac{4}{\sqrt{5}} \doteq 1.79$ miles.