MATH 135 – Calculus 1
"Derivative Practice"
October 23, 2019

Background

We have now seen the sum and product rules for derivatives. The goals for today are:

(1) To introduce another rule called the *quotient rule.*

(2) To practice using these rules, and

(3) To think about some of the information about a function that we can get from the derivative.

The quotient rule for derivatives says: If f, g are differentiable and $g(x) \neq 0$ then f/g is differentiable at x and

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2}$$

("bottom times derivative of top minus top times derivative of bottom, all over bottom squared"). For instance

$$\frac{d}{dx}\left(\frac{x^2+3}{x^4+x}\right) = \frac{(x^4+x)(2x) - (x^2+3)(4x^3)}{(x^4+x)^2} = \frac{-2x^5 - 4x^3 + 2x^2}{(x^4+x)^2}$$

(This could be simplified even farther by factoring x^2 out of the numerator and denominator. But notice that $x = 0$ is not in the domain of the original function, so it's not in the domain of the derivative either. Even though you could cancel x^2 between the top and bottom in the derivative, our function is not differentiable at $x = 0$.)

Questions

(1) Differentiate each of these with respect to the indicated variable. Note: you will want to think first about which rule(s) you need to apply, and then apply them. *Don't worry too much about simplifying your answers – any correct form is OK for this.*

(a) $f(x) = \dfrac{x^2 + e^x}{\sqrt{x}}$

Solution: Write $\sqrt{x} = x^{1/2}$. By the quotient rule:

$$\begin{aligned} f'(x) &= \frac{x^{1/2}(2x + e^x) - (x^2 + e^x)\frac{1}{2}x^{-1/2}}{(x^{1/2})^2} \\ &= \frac{3x^2 + 2xe^x - e^x}{2x^{3/2}} \end{aligned}$$

(Note that this function could also be rewritten as $f(x) = x^{3/2} + x^{-1/2}e^x$ and differentiated using the power, product, and exponential rules. That's somewhat easier!)

(b) $g(t) = e^t\left(1 + \dfrac{t^2}{1+t^2}\right)$

Solution: Use the product and quotient rules:

$$g'(t) = e^t\left(\frac{(1+t^2)(2t) - t^2(2t)}{(1+t^2)^2}\right) + e^t\left(1 + \frac{t^2}{1+t^2}\right).$$

(c) $h(z) = \dfrac{3}{z^{2/3}} - z(e^z + 4z)$

it Solution: It's easier to write the first part of this as a power instead of using the quotient rule:

$$h(z) = 3z^{-2/3} - z(e^z + 4z)$$

so

$$h'(z) = -2z^{-5/3} - z(e^z + 4) - (e^z + 4z) = -2z^{-5/3} - ze^z - e^z - 8z.$$

(2) Section 3.4 in our book builds on the way we motivated the study of derivatives by considering instantaneous velocities and slopes of tangent lines. If f is any function and $f'(a)$ exists, then we can think of $f'(a)$ as an *(instantaneous) rate of change* of f with respect to the variable in f, at a. The *units* of an instantaneous rate of change are always (units of f-values)/(units of the input variable in f). For instance, if we had a function $P(R)$ giving the electrical power (in units of watts) delivered to a device by a battery, as a function of the resistance of the device (in units of ohms), then the units of $P'(R)$ would be watts/ohm. So suppose we have a battery delivering power to a device with

$$P(R) = \frac{2.25R}{(R+.5)^2} = \frac{2.25R}{R^2 + R + .25}$$

where $R \geq 0$.

(a) What is the instantaneous rate of change of the power with respect to resistance when $R = 3$ ohms? (Give your answer with correct units.)

Solution: First we compute by the quotient rule:

$$P'(R) = \frac{(R^2 + R + .25)(2.25) - 2.25R(2R+1)}{(R^2 + R + .25)^2} = \frac{-2.25R^2 + .5625}{(R^2 + R + .25)^2}.$$

We want $P'(3) = \frac{-2.25\cdot 3^2 + .5625}{(3^2+3+.25)^2} \doteq -.13$ watts/ohm. (This indicates that the power is decreasing as the resistance increases near $R = 3$.)

(b) What is the power delivered to a device with $R = 5$ ohms? What is the instantaneous rate of change of the power with respect to resistance when $R = 5$ ohms? Give each answer with the correct units.)

Solution: The power is $P(5) \doteq .37$ watts. The rate of change of power with respect to resistance is $P'(5) \doteq -.06$ watts/ohm.

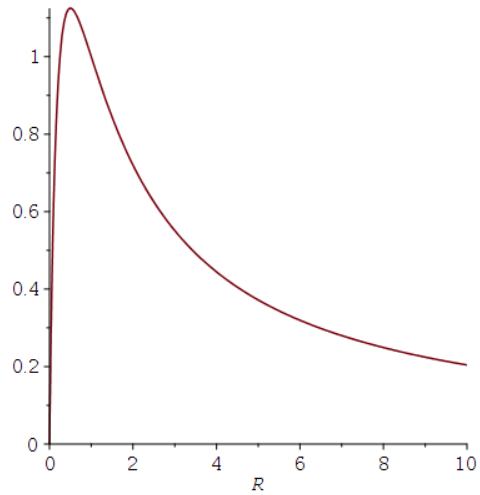

Figure 1: The graph $P = \frac{2.25R}{(R+.5)^2}$ for $0 \leq R \leq 10$

(c) Is the instantaneous rate of change of the power with respect to resistance ever equal to zero? What does that mean? Generate a sketch of the graph of $P(R)$ (R on the horizontal axis, P on the vertical axis) and show any points where $P'(R) = 0$.

Solution: Yes, the rate of change of power with respect to resistance is zero when

$$0 = P'(R) \Rightarrow -2.25R^2 - .5626 = 0 \Rightarrow R = \pm .5$$

(I'm using the simplified form of $P'(R)$ from part (a) to do this easily.) Since we only want $R > 0$, the positive root $R = .5$ is the only one. The graph of $P(R)$ for positive R indicates what is happening: