MATH 135 – Calculus 1
Applied Optimization Problems
November 22 and 25, 2019

Background

In today's video, we discussed some first examples of optimization or "max/min" problems. Many of these come from potentially realistic applications. A general strategy for solving these is:

- Step 1: Draw pictures illustrating several different possible solutions of the problem, if appropriate.
- Step 2: Identify which quantities are changing in your different solutions (the variables), and name them. Write down the quantity to be maximized or minimized, giving it a name too.
- Step 3: (If there is more than one variable), write down any relations between the variables, and use the relation to solve for all variables in terms of one of them.
- Step 4: Substitute from results of Step 3 into the function giving the quantity from Step 2. The goal here is to obtain a function of *only one* variable.
- Step 5: Find the critical points of the function from Step 4.
- Step 6. Classify critical critical as local maxima or local minima using First or Second Derivative Tests. If the variable is limited to an interval, determine the function values at the endpoints, and find the overall maximum or minimum as indicated in the problem. (Note: "largest, biggest, greatest, etc." in the statement of the problem usually means you are looking for a maximum value, while "smallest, least, cheapest, etc." usually indicates you are looking for a minimum.)
- Step 7. Find the maximum or minimum and write down the final answer. (And, of course, be sure you are answering the question that was asked!)

Today and next time, we want to practice using this on several examples.

Note: Unlike what has happened on the previous daily worksheets, I will ask you to hand in your work on this one by the end of the class on Monday, November 25 and I will grade it and return it on Monday, December 2. The score will be included in the Daily Quiz grade total, since we will not have a daily quiz to be completed for Monday November 25. Also, the third midterm exam *will cover* this material.

Discussion Questions

A) Daphnis Aipolos, a Greek goatherd, wants to fence in a rectangular pasture for his prize goats. He has 1600 meters of fencing to use, and he can use the straight-line section of the sea shore bordering his master's land as one of the boundaries of the pasture (the goats don't like the water and won't try to swim away in the sea). What dimensions should the pasture be to maximize the area enclosed for the goats?

Solution: Let x be the length of the side parallel to the sea shore, and let y be the length of the two sides perpendicular to the sea shore. If Daphnis uses all the fencing (and he clearly should to maximize the area enclosed!), then

$$2y + x = 1600 \text{ and hence } y = 800 - \frac{x}{2}.$$

The area is then

$$A = xy = x\left(800 - \frac{x}{2}\right) = 800x - \frac{x^2}{2}.$$

Then to find the maximum area we want to find the maximum of the function above on the interval $[0, 1600]$. The critical points are the roots of $A'(x) = 800 - x = 0$ so $x = 800$. Since $A''(800) = -1$, by the Second Derivative Test, there is a local and absolute maximum of $A(x)$ at $x = 800$. The dimensions of the pasture should be $x = 800$, $y = 400$, and the maximum area is $A(800) = 320000$ square meters.

B) A metal can with a top is to be manufactured to contain 30 cubic inches of "Soprano Family Jersey Special" pizza sauce. The can will have the shape of a right circular cylinder. Find the dimensions (height and radius of the cylinder) that will *minimize the cost of the can*, assuming the metal for the top costs 1 cent per square inch, but the metal for the bottom and the curved side wall of the can costs 2 cents per square inch because of a special acid-resistant coating used to increase the shelf-life of the pizza sauce. (The inside top of the can does not need the acid-resistant coating because the cans are never completely filled with the sauce!)

Solution: Let the radius and height of the cylinder be r, h. We want to maximize the cost, which is computed by adding the cost of the bottom, the cost of the side wall and the cost of the top:

$$C = (2)(\pi r^2) + (2)(2\pi rh) + (1)(\pi r^2) = 3\pi r^2 + 4\pi rh,$$

subject to the constraint that $V = \pi r^2 h = 30$. From the volume constraint,

$$h = \frac{30}{\pi r^2}$$

and then

$$C(r) = 3\pi r^2 + \frac{120}{r}.$$

We must have $r > 0$, of course. There is a critical point where

$$0 = C'(r) = 6\pi r - \frac{120}{r^2},$$

or $r = \sqrt[3]{\frac{20}{\pi}}$. This is a local and global minimum by the Second Derivative Test since

$$C''(r) = 10\pi + \frac{180}{r} > 0$$

for all $r > 0$. The dimensions of the cheapest can are

$$r = \sqrt[3]{\frac{20}{\pi}} \doteq 1.85 \text{ in and } h \doteq \frac{30}{\pi(1.85)^2} \doteq 2.79 \text{ in.}$$

C) For some species of birds, it takes more energy to fly over water than over land (maybe because over land they can make use of updrafts). A lesser tufted grebe (this is a species of bird) leaves an island 5 km from point A, the nearest point to the island on a long straight shore. The grebe's nest is at point B, 13 km along the shore from point A. If it takes 1.4 times as much energy to fly one km over water as it does to fly one km over land, where on the shoreline should the bird head first in order to *minimize the total energy* needed for the flight from the island to the nest.

Solution: Let x be the distance along the shore where the bird first makes land. We clearly want $x > 0$ since taking $x < 0$ means that the part of the trip over land will be longer and take more energy. Then the distance over water is $d_w = \sqrt{25 + x^2}$. The distance over land is $d_\ell = 13 - x$. The energy cost is proportional to

$$E(x) = 1.4d_w + d_\ell = 1.4\sqrt{25 + x^2} + 13 - x.$$

This has a critical point at the solution of

$$0 = E'(x) = \frac{1.4x}{\sqrt{25 + x^2}} - 1,$$

or

$$\sqrt{25 + x^2} = 1.4x,$$

so $25 + x^2 = 1.96x^2$ or $x = \sqrt{25/.96} \doteq 5.1$. This is a local minimum by the First Derivative Test since $E'(x) < 0$ for $x < \sqrt{25/.96}$ and $E'(x) > 0$ for $x > \sqrt{25/.96}$.

D) Northern Iowa State Agricultural and Veterinary Junior College is building a new running track for their prize-winning track team – the "Flying Farmers." The track is to be the perimeter of a region obtained by putting two semicircles on opposite ends of a rectangle, and that perimeter should be 440 yards in length. Due to budget problems, the administration has decided to grow sweet corn in the area enclosed by the track and sell it to a local grocery store to generate some extra revenue. Determine the dimensions to build the track in order to *maximize the area for growing corn.* (*Careful*: This is a slightly "tricky" one in that the maximum is actually at an endpoint of the interval of possible values of one of the "obvious" variables.)

Solution: Suppose x, y are the sides of the rectangle, and the semicircular areas are to be placed on the sides of length x. Then the radii of the semicircles are $x/2$ and the total area is found by adding the area of the rectangle and the two semicircles:

$$A = xy + \pi(x/2)^2 = xy + \frac{\pi x^2}{4}.$$

The perimeter must be 440 yards, so $2y + \pi x = 440$, and hence $y = 220 - \frac{\pi x}{2}$. We substitute this into the formula for the area to obtain

$$A(x) = x\left(220 - \frac{\pi x}{2}\right) + \frac{\pi x^2}{4} = 220x - \frac{\pi x^2}{4}.$$

We want to maximize this on the interval $[0, 440/\pi]$ (the value $x = 0$ corresponds to a straight line track with no semicircles; the value $x = 440/\pi$ corresponds to a circular track with no rectangle in the middle, since then $y = 0$). The tricky thing about this problem is that there is a critical point:

$$A'(x) = 220 - \frac{\pi x}{2} = 0 \text{ when } x = \frac{440}{\pi}.$$

It is a local and global maximum of the area function since $A''(x) = \frac{-\pi}{2} < 0$. This means that the maximum occurs for the circular track with no rectangular portion.