MATH 135 – Calculus 1
The Derivative Function
October 19, 2016

Background

On Monday, we were working with the *derivative* of a function f at x:

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h},$$

provided the limit exists. One interpretation of the value $f'(a)$ is

$$f'(a) = \text{slope of tangent line to } y = f(x) \text{ at } x = a.$$

Today, we want to understand how $f'(a)$ can fail to exist at some points, and also begin to understand how the functions $f(x)$ and $f'(x)$ are related to each other.

Questions

(1) First consider $f(x) = x^{1/3}$ at $x = 0$.

(a) What happens if we apply the limit definition to compute $f'(0)$ for this function?

(b) The graph $y = x^{1/3}$ can be obtained by reflecting $y = x^3$ across the line $y = x$ (why?) Use this to sketch the graph $y = x^{1/3}$. What's happening at $x = 0$? Hint: The graph $y = x^{1/3}$ *does have a tangent line at* $x = 0$, but what can you say about that line? Why isn't $f'(0)$ defined?

(2) Now consider

$$f(x) = x|x-2| = \begin{cases} 2x - x^2 & \text{if } x \le 2 \\ x^2 - 2x & \text{if } x > 2 \end{cases}$$

(a) Sketch the graph $y = f(x)$ for $0 \le x \le 4$. What happens at $x = 2$? Why might we expect that there would be a problem defining $f'(2)$? (Does this graph seem to have a reasonable tangent line at $x = 2$?)

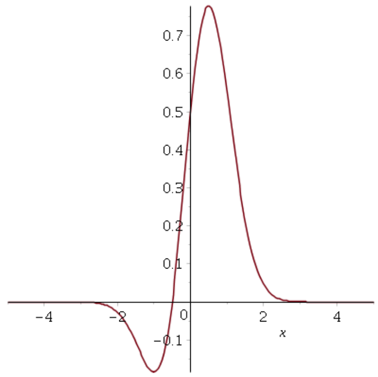
(b) Determine the two one-sided limits

$$\lim_{h\to 0^+} \frac{f(2+h) - f(2)}{h} \quad \text{and} \quad \lim_{h\to 0^-} \frac{f(2+h) - f(2)}{h}$$

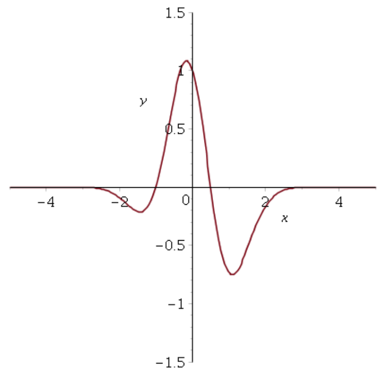
This should reinforce what you were saying in part (a)

(3) Now look at plots A,B on the back of this page. One is a graph $y = f(x)$ and the other is the graph $y = f'(x)$ for the same function. Which graph is $y = f(x)$ and which is $y = f'(x)$? How can you tell? (Hints: Look at plot A. Over which intervals does that function have tangent lines with positive slope? Over which intervals does that function have tangent lines with negative slope? Where does that graph have horizontal tangent lines (slope 0)? You'll need to estimate the endpoints from the axis scales.) Now look at plot B. Over which intervals does that function have positive values? Over which intervals does that function have negative values? slope? Where does that function equal zero? (You'll need to estimate the endpoints from the axis scales.))

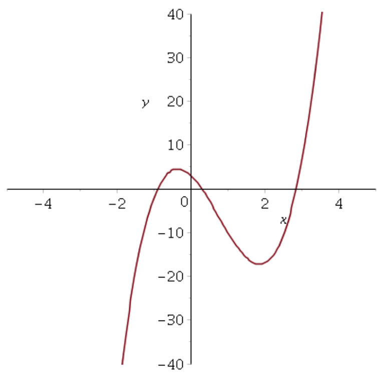
Figure 1: Plots for Questions 3 and 4



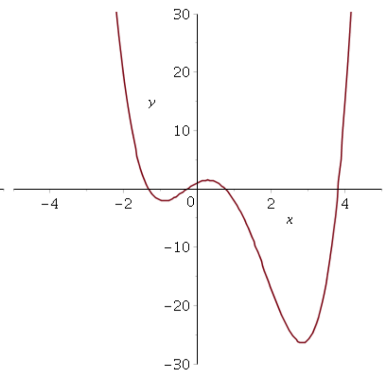
(a) Plot A



(b) Plot B



(c) Plot C



(d) Plot D

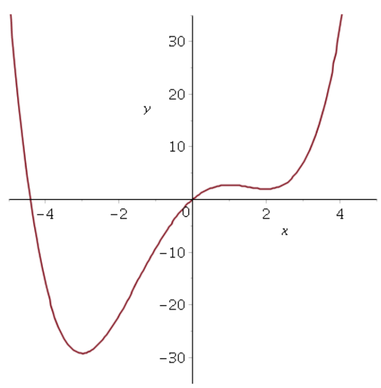


Figure 2: Plot E – for Question 5

(4) Now look at plots C,D. Again one is $y = f(x)$ and one is $y = f'(x)$. Which is which? How can you tell?

(5) Now refer to plot E. This is $y = f(x)$. Draw a "qualitative" plot of $y = f'(x)$. This means: Don't worry about exact values of $f'(x)$, but generate a plot of $y = f'(x)$ showing the intervals where $f'(x) > 0$, where $f'(x) < 0$ and the points where $f'(x) = 0$.