MATH 135 – Calculus 1
Logarithm and Exponential Functions
September 13, 2019

Background

Logarithm functions are the inverse functions of exponential functions $f(x) = b^x$. It may also help to think of what my high school math teacher Mr. Brennan[1] said, *"a logarithm is an exponent."* Of course, this is another way of saying that the $\log_b$ function is the inverse function of the b^x exponential:

$$y = \log_b x \Longleftrightarrow x = b^y,$$

so $\log_b(b^y) = y$ for all real y, and $b^{\log_b(x)} = x$ for all $x > 0$. The most important logarithm function for us will be the *natural logarithm* $\ln(x)$, which is the same as $\log_e(x)$ where $e \doteq 2.71828\cdots$ is (for now) a somewhat mysterious number(!) From the properties of exponents, (see the table on page 41 of our text), we get the key properties of the ln function:

(1) $\ln(x_1 x_2) = \ln(x_1) + \ln(x_2)$,

(2) $\ln\left(\frac{x_1}{x_2}\right) = \ln(x_1) - \ln(x_2)$, and

(3) $\ln\left(A^B\right) = B\ln(A)$,

and similarly for $\log_b$ for any $b > 0$. Today, we want to practice using these properties of the logarithm functions to simplify expressions and solve equations.

Questions

1 *Moore's law* is the observation that the number of transistors in the densest integrated circuits (like those used in computer hardware) *doubles approximately every two years*. The observation is named after Gordon E. Moore, the co-founder of Intel and Fairchild Semiconductor. Moore wrote a paper in 1965 describing a doubling *every year* in the number of components per integrated circuit and projected this rate of growth would continue for at least another decade. In 1975, looking forward to the next decade, he revised the forecast to doubling *every two years*. This prediction proved very accurate for several decades, and the law was used in the semiconductor industry to guide long-term planning and to set targets for research and development. Advancements in digital electronics are strongly linked to Moore's law: memory capacity, sensors and even the number and size of pixels in digital cameras and cell phones. (In other words, your "smart phones" would not be possible without it!)

In this question, we want to see how Moore's Law leads to something closely related to one of the exponential functions.

[1] He was a *total nerd*. (Maybe you would say I take after him and I am too!)

(a) What we'll do first today is to take two actual data points and find a function of the form $f(x) = c \cdot b^x$ that agrees with them. In 1990, the Intel 80486 CPU chip (one industry standard at the time – we had lots of PC's at Holy Cross with those CPUs!) contained about 1,000,000 transistors. In 2005, the AMD K8 CPU chip contained about 100,000,000 transistors. To simplify things, let's say x represents the number of years after 1990, so $1990 \leftrightarrow x = 0$ and $2005 \leftrightarrow x = 15$. There is exactly one function of the form $f(x) = cb^x$ whose graph contains the points $(0, 1000000)$ and $(15, 100000000)$. To find the c and b that work, first substitute these x and y values to get

$$\begin{aligned} 1000000 &= cb^0 \quad (1) \\ 100000000 &= cb^{15}. \quad (2) \end{aligned}$$

To finish this off, use the first equation to solve for c, then substitute your value for c into the second equation and solve for b. Use your values for b, c to write a function that represents the number of transistors in the densest circuits as a function of time.

(b) Now, we want to know how close this is to saying "the number of transistors doubled approximately every two years." If we started from the known value $f(0) = 1000000$ (the number of transistors in the 80486 chip from 1990), then the number after 2 years should be 2×1000000, then after 4 years the number should be

$$2 \times 2 \times 1000000 = 4 \times 1000000,$$

the number after 6 years should be

$$2 \times 2 \times 2 \times 1000000 = 8 \times 1000000,$$

and so forth. In general, suppose x is the number of years after 1990, then $x/2$ would give the number of 2-year periods between 1990 and $1990 + x$, and hence the *number of doubling factors*. Find a formula that would give the predicted number of transistors as a function of x. (Hint: The number would have doubled $x/2$ times to match this pattern).

(c) Explain how to rewrite $2^{x/2}$ in the form b^x for some b (you'll need to recall and use a rule of exponents to do this). How does this value compare with the b you found in part (a) of the question?

(d) To be clear, "Moore's law" is an just observation or projection of what has happened in the development of computer technology; it is obviously not a physical or natural law, and it cannot continue to hold true indefinitely into the future. Why not? Explain in your own words by thinking about the formula you developed earlier.

2) Simplify and compute the value. (Do these *without using a calculator*!)

(a) $\log_2 \left(\frac{1}{128}\right)$

(b) $\log_5(125)$

(c) $\ln(e^4) + \ln\left(\frac{1}{e^5}\right)$

3) Solve each of these equations for the unknown by using properties of logarithms. Get an exact answer (expressed using logarithms), then find a decimal approximation using a calculator.

   (a) $e^{5x+1} = 2$.

   (b) $2^{2x} = 3^{3x-5}$.

4) (a) Explain why $f(x) = e^{2(x-1)} + 1$ has an inverse function. (Hint: What is the property that says a function has an inverse function? Sketch the graph $y = e^{2(x-1)} + 1$ to see that this function "has it.")

   (b) Find a formula for f^{-1} for the function from part (a). Sketch the graph $y = f^{-1}(x)$ on the same axes as you had in part (a).