MATH 135 – Calculus 1
The Intermediate Value Theorem
October 7, 2019

Background

It's our lucky day, because today we get to discuss some *beautiful mathematics*(!) The Intermediate Value Theorem (IVT) is the following statement:

Theorem 1 (IVT) *Let $f(x)$ be a function that is continuous at every x in a closed interval $[a, b]$. Then for every M between and $f(a)$ and $f(b)$, there exists at least one $x = c$ in the interval where $f(c) = M$.*

What this is saying in intuitive terms is that the graph of a function that is continuous at every point of a closed interval *is one unbroken curve* that can be drawn without lifting your pen or pencil from the paper. In the process it goes through every y-value between the y-coordinates of the endpoints $(a, f(a))$ and $(b, f(b))$. This statement has some important and surprising consequences!

Questions

(1) An important consequence: The IVT implies that many equations have solutions(!) For instance, if $f(x)$ is a continuous function on an interval $[a, b]$ and $f(a) \cdot f(b) < 0$ (that is, $f(a)$ and $f(b)$ *have opposite signs*), then $f(x) = 0$ has a root $x = c$ somewhere in (a, b). (Note: the root cannot be at one of the endpoints, since then $f(a) \cdot f(b) = 0$. But we are assuming that product is < 0.)

   (a) Look at the particular cubic $P(x) = x^3 - 3x^2 + 5x - 7$ first. Find a real number a where $P(a) < 0$ and another $b > a$ where $P(b) > 0$. Explain why $P(x)$ is continuous at all x, then use the IVT to argue that $P(x) = 0$ has a root in your interval $[a, b]$.

   (b) Now consider a general cubic polynomial $P(x) = Ax^3 + Bx^2 + Cx + D$. Explain why $P(x) = 0$ must have at least one real root. (Hint: since $A \neq 0$ we can divide both sides of the equation $Ax^3 + Bx^2 + Cx + D = 0$ by A to get an equation of the form $x^3 + \beta x^2 + \gamma x + \delta = 0$, where $\beta = B/A$, $\gamma = C/A$, etc. Explain why, no matter what β, γ, δ are, you can always find a really big $x = b > 0$ where $b^3 + \beta b^2 + \gamma b + \delta > 0$. Then apply the IVT and argue that $P(x) = 0$ has a root in $[a, b]$.)

   (c) The same reasoning will apply to any polynomial $P(x)$ satisfying a particular condition on its *degree*, the highest power of x appearing with a nonzero coefficient. What is that condition?

   (d) Show that $s(x) = x^6 - 8x^4 + 10x^2 - 1 = 0$ has *six* real roots, using the IVT, by finding 6 different intervals $[a, b]$ where $s(a) \cdot s(b) < 0$. You can also find approximations to the roots with a graphing calculator, but that is not required.

(2) Another important consequence. Let f be continuous function on the interval $[0, 1]$ and suppose that $0 \leq f(x) \leq 1$ for all x in $[0, 1]$. Explain why $f(x) = x$ for some x in $[0, 1]$. (Hint: Draw a picture to see why this is expected, then consider the function $f(x) - x$ and apply the IVT.

(3) A related, but surprising, consequence. Take any map and draw a circle on it anywhere. (See Figure 6 on page 103 of our textbook, for example.) I claim that the there are two distinct points on the earth corresponding to points on that circle where the temperature right now

is *exactly the same*. Here's the idea: If we put in a coordinate system with the center of the circle at $(0, 0)$, then we can measure the counterclockwise angle θ from the positive x-axis to any diameter of the circle. Let $f(\theta)$ be *the difference between the temperatures at the two endpoints of the diameter*. In other words, if A and B are the two endpoints as in the figure, then

$$f(\theta) = (\text{temperature at } A) - (\text{temperature at } B).$$

It is reasonable to assume that $f(\theta)$ is a continuous function of θ (Why?). Then consider what happens as θ increases from 0 to π radians. How are the values $f(0)$ and $f(\pi)$ related? What does the IVT tell you about f on the interval $[0, \pi]$?