MATH 135 – Calculus 1
Extreme Values
November 11, 2019

Background

Recall from today's video:

- A continuous function on a closed interval attains a *maximum value* and a *minimum value* on that interval.
- Those maximum and minimum values are attained either at endpoints of the interval or at interior points – nothing deep going on there, just logic(!)
- If the maximum or minimum value is attained at a point $x = c$ *other than an endpoint*, then c must be a *critical point* of f – either a solution of $f'(c) = 0$, or else a place where $f'(c)$ does not exist.
- So to find the maximum and minimum value of a continuous function of a continuous $f(x)$ on a closed interval $[a, b]$, we can:
  - (i) Compute $f'(x)$ and find all critical points c in $[a, b]$.
  - (ii) Compute $f(a), f(b)$, and $f(c)$ for all critical points found in the first step.
  - (iii) Then the maximum value will be the largest of the numbers found in the previous step and the minimum value will be the smallest of those numbers.

Questions

For each of the following functions,

(i) determine all critical points in the given interval,

(ii) compute the values of f at the critical points in the interval, and compute the values at the endpoints of the interval,

(iii) determine the maximum and minimum values of f on the interval.

This is the process sketched above(!)

1. $f(x) = x^3 - 12x^2 + 21x$ on $[0, 11]$.

   Solution: $f'(x) = 3x^2 - 24x + 21 = 3(x - 7)(x - 1)$, so f has two critical points $x = 1, 7$; both are in the interval $[0, 11]$. (Note: f and f' are polynomial functions, so they are defined for all real x.) Comparing the values

$$\begin{aligned} f(0) &= 0 \\ f(11) &= 110 \leftarrow \text{ maximum} \\ f(1) &= 10 \\ f(7) &= -98 \leftarrow \text{ minimum} \end{aligned}$$

2. $f(x) = (x^2 + 2x)e^{-x}$ on $[1, 5]$

   Solution: $f'(x) = (2 - x^2)e^{-x}$. Thus $f'(x) =$ at $x = \pm\sqrt{2}$, but $x = \sqrt{2}$ is the only critical point in the interval $[1, 5]$. As in (1), both f and f' are defined for all real x. Comparing the values

$$\begin{aligned} f(1) &\doteq 1.104 \\ f(\sqrt{2}) &\doteq 1.174 \leftarrow \text{maximum} \\ f(5) &\doteq .0236 \leftarrow \text{minimum} \end{aligned}$$

3. $f(x) = 5\tan^{-1}(x) - x$ on $[-5, 5]$.

   Solution: We have

$$f'(x) = \frac{5}{1+x^2} - 1 = \frac{4-x^2}{1+x^2}$$

   This (and $f(x)$) is defined for all real x. The critical points are $x = \pm 2$. Comparing the values

$$\begin{aligned} f(-5) &\doteq -1.867 \\ f(-2) &\doteq -5.536 \leftarrow \text{minimum} \\ f(2) &\doteq 3.536 \leftarrow \text{maximum} \\ f(5) &\doteq 1.867 \end{aligned}$$

4. $f(x) = (x - x^2)^{2/3}$ on $[0, 2]$. (Note: Careful on this one – you should find critical points where $f'(x)$ does not exist.)

   Solution: By the Chain Rule have $f'(x) = \frac{2}{3}(x - x^2)^{-1/3}(1 - 2x)$. This is undefined at $x = 0, 1$ and it is equal to zero at $x = 1/2$. These are all critical points. In addition, $x = 0$ is one of the endpoints of the interval. Comparing the values:

$$\begin{aligned} f(0) &= 0 \leftarrow \text{minimum} \\ f(1/2) &\doteq .397 \\ f(1) &= 0 \leftarrow \text{minimum} \\ f(2) &\doteq 1.587 \leftarrow \text{maximum} \end{aligned}$$