MATH 135 – Calculus 1
Discussion Days on Limits and Continuity
September 23-25, 2019

Background, part A

Recall that we saw last time that a number of basic limits can be computed by breaking a function up into simpler "pieces" via sums, products, quotients, powers, etc. and using the Limit Laws. One important thing to realize is that even if a particular limit is not one we can do with the Limit Laws directly, *might become possible to apply them* after some algebraic rearrangement or manipulation. Here are some examples.

(1) Consider $\lim_{x\to 2} \frac{x^2-4x+4}{x^2-5x+6}$.

   (a) Generate an accurate sketch of the graph $y = \frac{x^2-4x+4}{x^2-5x+6}$ using a graphing calculator. Can you guess what the limit should be that way?

   (b) Why don't the basic Limit Laws (the quotient law in particular) apply here?

   (c) What if you factor the numerator and the denominator? Can you see what the limit is now after cancelling?

(2) Consider $\lim_{x\to 0} \frac{\sqrt{x^2+9}-3}{x}$.

   (a) Generate an accurate sketch of the graph $y = \frac{\sqrt{x^2+9}-3}{x}$ using a graphing calculator.

   (b) Why don't the basic Limit Laws (the quotient law in particular) apply here?

   (c) What if you multiply the top and the bottom of the fraction by $\sqrt{x^2+9}+3$ and simplify? Can you see what the limit is now after cancelling what you can?

Background, part B

We say $f(x)$ is continuous at $x = c$ if

$$\lim_{x\to c} f(x) = f(c).$$

If $\lim_{x\to c} f(x)$ does not exist, or if it does exist but is different from $f(c)$, then we say $f(x)$ has a *discontinuity* at c.

For example, the function in Figure 1 has a discontinuity at $x = 2$. This sort of discontinuity, where $\lim_{x\to c^+} f(x)$ and $\lim_{x\to c^-} f(x)$ both exist but are *different* is called a *jump discontinuity.*

Another type of discontinuity is shown in Figure 2 (top of page 3) at the points $x = \pm 1$ where the graph has vertical asymptotes, also known as *infinite discontinuities.* Note that we could say

$$\lim_{x\to 1^+} f(x) = +\infty, \lim_{x\to 1^-} f(x) = -\infty, \lim_{x\to -1^+} f(x) = +\infty, \lim_{x\to 1^-} f(x) = -\infty$$

to describe the way the graph is approaching the vertical asymptotes on either side.

People often say that continuity at c means intuitively that the graph has no "hole," jump, or vertical asymptote at $x = c$ and so the part of the graph near $x = c$ can be drawn as one unbroken curve without lifting your pencil or pen from the paper. This is somewhat useful intuitively, but you should not take it too literally. There are "strange" functions that are continuous at some c where that physical description is not true at all! For instance, the "strange function" defined piecewise by

$$f(x) = \begin{cases} x & \text{if } x \text{ is a rational number} \\ -x & \text{if } x \text{ is an irrational number} \end{cases}$$

is actually continuous at $c = 0$ since $\lim_{x\to 0} f(x) = 0 = f(0)$. You should be aware also that holes, jumps, and vertical asymptotes is far from a complete list of all the possible types of *discontinuities*.

Questions

(1) Consider the function

$$f(x) = \begin{cases} x^2 + 3 & \text{if } x < 1 \\ -x + 10 & \text{if } 1 \le x \le 2 \\ -x^2 + 6 & \text{if } x > 2, x \ne 3 \\ 10 & \text{if } x = 3. \end{cases}$$

Generate (by hand) a plot of the graph $y = f(x)$ that is accurate enough to let you answer the following questions. At each of the given x values, is f continuous? Why or why not? (That is, if f is not continuous there, what part of the definition of continuity fails?) And if f is not continuous there could you redefine the value $f(x)$ to make a new continuous function? (That is, is the discontinuity *removable*?)

(a) $x = 1$
(b) $x = 2$
(c) $x = 3$
(d) $x = 4$

(2) Which of the following quantities would you expect to be represented by *continuous* functions of time and which would have one or more discontinuities? Explain.

(a) The price of a Starbucks Venti Mocha Latte over a period of several years.
(b) Your distance from home during a car trip.
(c) The number of students enrolled as Holy Cross undergraduates over a period of four years.
(d) The temperature reading at the top of Mount Washington in New Hampshire. (Does it matter what type of thermometer you use for this? What if you use an old-fashioned mercury thermometer and you can read the temperature to arbitrary precision? What if you use a digital thermometer that only gives readings in tenths of degrees?)

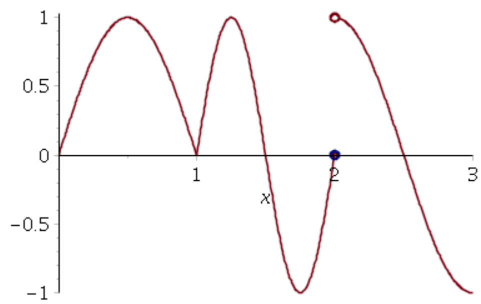

Figure 1: A function with a discontinuity at $x = 2$.

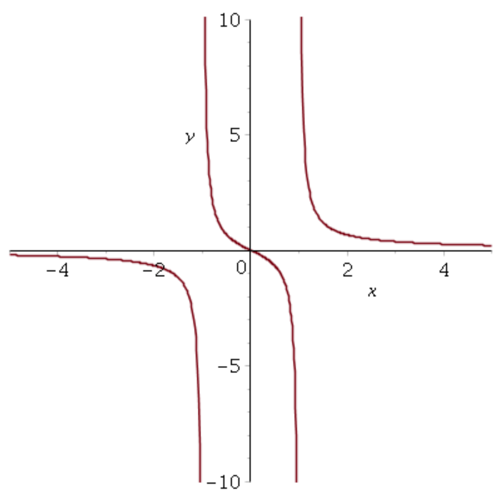

Figure 2: A function with discontinuities at $x = \pm 1$.

(3) The function $f(x) = \dfrac{x^2}{16 - x^2}$ has two points of discontinuity of the form $x = \pm c$. Say what they are and determine

$$\lim_{x \to c^-} f(x), \lim_{x \to c^+} f(x), \lim_{x \to -c^-} f(x), \lim_{x \to -c^+} f(x)$$

(4) In 2009, the Federal income tax T on on incomes x up to \$82,250 was determined by the formula

$$T(x) = \begin{cases} 0.10x & \text{if } 0 \le x \le 8350 \\ 0.15x - 417.50 & \text{if } 8350 \le x < 33950 \\ 0.25x - 3812.50 & \text{if } 33950 \le x < 82250. \end{cases}$$

Does T have any discontinuities? Why or why not? Might it be advantageous in some situations to earn *less* money? Explain.