MATH 135 – Calculus 1
Second Derivative and Concavity
November 15, 2019

Background

We say f (or the graph $y = f(x)$) is *concave up* on an interval if f' is increasing on that interval, and similarly, f or its graph is *concave down* of f' is decreasing on that interval. Combined with our results from last time, this says:

- If $f''(x) > 0$ on an interval, then f or its graph is concave up on that interval
- If $f''(x) < 0$ on an interval, then f or its graph is concave down on that interval
- A point $(c, f(c))$ on the graph of f where the concavity changes is called a *point of inflection* of f.

The notion of concavity can also be used to state a second method for determining whether critical points are local maxima or local minima, called the Second Derivative Test:

Theorem 1 *(Second Derivative Test) Let f be differentiable on some open interval containing a critical point c. In addition, assume $f''(c)$ exists.*

(a) If $f''(c) > 0$, then $f(c)$ is a local minimum

(b) If $f''(c) < 0$, then $f(c)$ is a local maximum

(c) If $f''(c) = 0$, there is no conclusion.

In the last case here, f could have either a local maximum or a local minimum, or neither, so no conclusion is possible. *Technical Comment*: In the other cases, the intuition is that f' should be increasing or decreasing on an interval containing c depending on the sign of $f''(c) = (f')'(c)$, so that (a) corresponds to a case where the graph is concave up at c and (b) corresponds to a case where the graph is concave down at c. This would follow, for instance, if we knew (in addition) that f'' was continuous on some interval containing c. But the conclusion of the Theorem is valid even without that extra continuity hypothesis, as is shown in Exercise 67 in Section 4.4.

Questions

1. Consider $f(x) = x^2 e^{-x}$.

   (a) Compute $f'(x)$ and find all critical points.

   Answer: $f'(x) = (-x^2 + 2x)e^{-x}$. This exists for all real x. Now $e^{-x} \neq 0$ for all real x, so the critical points come by solving $-x^2 + 2x = x(-x + 2) = 0$, so $x = 0$ and $x = 2$ are the critical points.

   (b) Determine the sign of $f'(x)$ on each interval between successive critical points, and use that to classify the critical points as local maxima or local minima by the First Derivative Test.

Answer: On $(-\infty, 0)$ taking $x = -1$ we see $f'(-1) = -3e < 0$. Hence f is *decreasing* on $(-\infty, 1)$. On $(0, 2)$, taking $x = 1$, we see $f'(1) = e^{-1} > 0$, so f is *increasing* on $(0, 2)$. Finally on $(2, \infty)$, $f'(3) = -3e^{-3} < 0$, so f is *decreasing* on $(2, \infty)$. By the First Derivative Test, this says that f has a *local minimum* at $x = 0$ and a *local maximum* at $x = 2$.

(c) Now compute $f''(x)$ and check your answers in (b) by using the Second Derivative Test.

Answer: $f''(x) = (x^2 - 4x + 2)e^{-x}$. At the first critical point $x = 0$, we have $f''(0) = 2 > 0$ so we see (in a different way) that f has a local minimum there. At the second critical point $x = 2$, $f''(2) = -4e^{-2} < 0$. Hence f has a local maximum there. This also agrees with what we saw above in part (b).

(d) Determine all points of inflection of f.

Answer: The points of inflection are the x where $f''(x)$ changes sign. This happens here when $f''(x) = 0$, or at the roots of $x^2 - 4x + 2 = 0$. By the quadratic formula,

$$x = \frac{4 \pm \sqrt{8}}{2} = 2 \pm \sqrt{2}.$$

2. Consider the graph $f(x) = x^3 - 3x^2 + 2x$ on the interval $[-1, 3]$ (the plot is on the back of this sheet). Find the intervals where f is concave up and the intervals where f is concave down. How many points of inflection are there on this graph and where are they located?

Answer: $f'(x) = 3x^2 - 6x + 2$, so $f''(x) = 6x - 6$. This is $= 0$ and changes sign at $x = 1$ ($f''(x) < 0$ for $x < 1$ and $f''(x) > 0$ for $x > 1$). So $y = f(x)$:

- is concave down on $(-\infty, 1)$,
- is concave up on $(1, \infty)$, and
- $(1, 0)$ is the only point of inflection.

3. Repeat question 2 for $f(x) = 2x^4 - 3x^2 + 2$.

Answer: $f'(x) = 8x^3 - 6x$ and $f''(x) = 24x^2 - 6$. This changes sign at $x = \pm\sqrt{1/4} = \pm\frac{1}{2}$. $y = f(x)$:

- is concave up on $\left(-\infty, -\frac{1}{2}\right)$ and $\left(\frac{1}{2}, \infty\right)$.
- is concave down on $\left(-\frac{1}{2}, \frac{1}{2}\right)$,
- and has inflection points at $x = \pm\frac{1}{2}$.

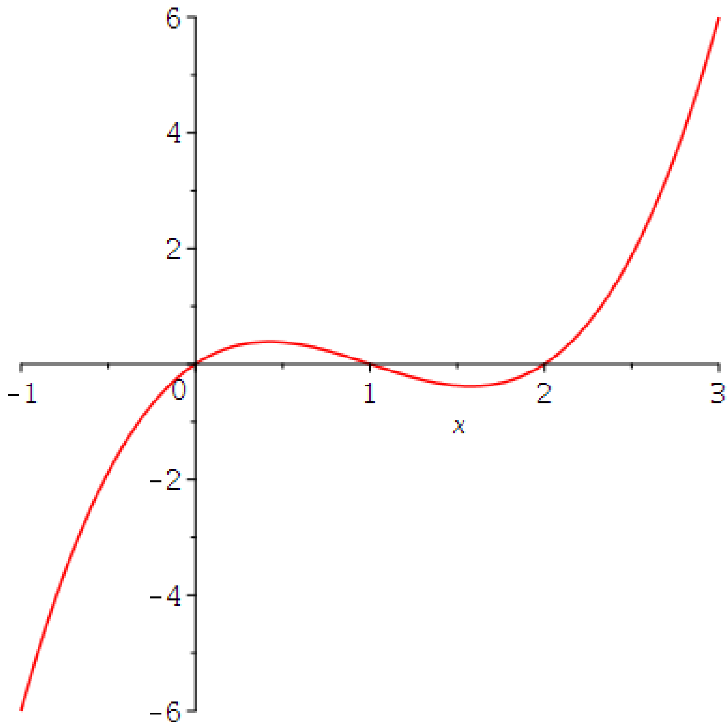

Figure 1: Plot for question 2