MATH 135 – Calculus 1
The Derivative Chain Rule
November 4, 2019

Background

Our next major derivative short-cut rule is *one of the most important.* This rule, called the *Chain Rule* allows us to differentiate functions that are built up by *composition.* Here's what it says: If g is differentiable at x and f is differentiable at $g(x)$, then the composition $(f \circ g)(x) = f(g(x))$ is differentiable at x and

$$\frac{d}{dx} f(g(x)) = f'(g(x))g'(x).$$

In words: The derivative of the composition is *the derivative of the outside function (i.e. f'), with $g(x)$ "plugged in," times the derivative of g.* For example, the function

$$h(x) = \sqrt{x^2 + 4x + 9}$$

is the composition $f(g(x))$ where $f(x) = \sqrt{x}$ and $g(x) = x^2 + 4x + 9$ is "plugged in." The Chain Rule says the derivative of h will be given by computing $f'(x) = \frac{1}{2\sqrt{x}}$ (do you see where that comes from?), plugging g into $f'(x)$, then multiplying by $g'(x)$:

$$h'(x) = \frac{1}{2\sqrt{x^2 + 4x + 9}} \cdot (2x + 4) = \frac{x + 2}{\sqrt{x^2 + 4x + 9}}.$$

Today's class will be devoted to understanding and practicing this rule. We'll continue and use this a different way next time.

Questions

For each function, identify an $f(x)$ and $g(x)$ such that the given function is the composition $f(g(x))$. Then apply the Chain Rule and compute the derivative:

(a)

$$h(x) = e^{3x+1}.$$

Answer: $h'(x) = 3e^{3x+1}$. $f(x) = e^x$ and $g(x) = 3x + 1$. The 3 in front is the derivative of $g(x)$. which is the "inside" function in this case.)

(b)

$$h(x) = \frac{1}{(x^4 + 5x^2 + 1)^{3/2}}.$$

Answer: $h'(x) = \dfrac{-3}{2}(x^4 + 5x^2 + 1)^{-5/2} \cdot (4x^3 + 10x)$. Here the $-3/2$ power is the "outside function" $f(x)$ and $g(x) = x^4 + 5x^2 + 1$

(c)

$$h(x) = \sin(\cos(x) + x).$$

Answer: $h'(x) = \cos(\cos(x) + x) \cdot (-\sin(x) + 1)$. Here $f(x) = \sin(x)$ and $g(x) = \cos(x) + x$.

(d)
$$h(x) = (\tan(x) + 4x)^3.$$

Answer: $f(x) = x^3$ and $g(x) = \tan(x) + 4x$, so the derivative is $h'(x) = 3(\tan(x) + 4x)^2 \cdot (\sec^2(x) + 4)$

(e) Sometimes we need to use the Chain Rule more than once (if the function we're looking at is "several composition layers deep" like
$$h(x) = \cos^2(4x^3 + 2) = (\cos(4x^3 + 2))^2.$$

Note that this is $f(g(x))$ with $f(x) = x^2$ and $g(x) = \cos(4x^3 + 2)$. But $g(x)$ is also a composition, so you'll need to use the Chain Rule again to find $g'(x)$. With these hints, find $h'(x)$.

Answer: $h'(x) = 2\cos(4x^3 + 2) \cdot (-\sin(4x^3 + 2)) \cdot 12x^2 = -24x^2 \cos(4x^3 + 2) \sin(4x^3 + 2)$