MATH 134 – Calculus with Fundamentals 2
Practice on L'Hopital's Rule
April 5, 2018

Background

L'Hopital's Rule is a method for determining whether indeterminate form limits such as $0/0$ limits or ∞/∞ limits exist and computing them if they do. It can also be applied in other indeterminate cases such as $\infty \cdot 0$ and 1^∞ by using additional algebraic maneuvers and/or properties of logarithms. The basic statement is this:

[L'Hopital's Rule] *If* $\frac{f(x)}{g(x)}$ *is* $0/0$ *or* ∞/∞ *as* $x \to c$ *(*c *can be finite or infinite, the limits can be one- or two-sided) and*

$$\lim_{x\to c} \frac{f'(x)}{g'(x)} = L$$

exists, then

$$\lim_{x\to c} \frac{f(x)}{g(x)} = L$$

as well.

Note: We are *not* applying the Quotient Rule to compute the derivative of $\frac{f(x)}{g(x)}$ here; L'Hopital's Rule uses the limit of the quotient of the derivatives of the numerator and denominator separately: $\frac{f'(x)}{g'(x)}$.

Questions

Use L'Hopital's Rule as indicated to compute the following limits

(1) $\lim_{x\to 1} \frac{x^3-1}{\sqrt{x}-1}$

(2) $\lim_{x\to\infty} \frac{\ln(x)}{x}$

(3) For $\infty \cdot 0$ forms $\lim_{x\to c} f(x) \cdot g(x)$, note that rewriting

$$f(x) \cdot g(x) = \frac{f(x)}{1/g(x)}$$

converts to a ∞/∞ form (if $\lim_{x\to c} g(x) = 0$ and $\lim_{x\to c} f(x) = \infty$). Hence you can apply L'Hopital to the rearranged form. Apply this idea to compute

$$\lim_{x\to 0^+} x\ln(x)$$

(4) If $\lim_{x\to c} f(x) = 1$ and $\lim_{x\to c} g(x) = \infty$, then $\lim_{x\to c} f(x)^{g(x)}$ is also an indeterminate form called a 1^∞ form. The standard approach for these is to take natural logs and use the fact that

$$\ln\left(f(x)^{g(x)}\right) = g(x)\ln(f(x))$$

The limit $\lim_{x\to c} g(x)\ln(f(x))$ is then an $\infty\cdot 0$ form and we can proceed as in (3). If

$$\lim_{x\to c} g(x)\ln(f(x)) = L$$

then the continuity of ln implies $\lim_{x\to c} f(x)^{g(x)} = e^L$. Apply this approach to compute

$$\lim_{x\to 0}(1+x)^{1/x}$$