4, 4, 8, 4 total = 20

MATH 134 Problem Set 6, part B solutions

§7.5 / 59. By partial fractions, when $a \neq b$

$$\frac{1}{(x-a)(x-b)} = \frac{A}{x-a} + \frac{B}{x-b}$$

so $1 = A(x-b) + B(x-a)$

For $x = a$, we get $1 = A(a-b)$ so $A = \frac{1}{a-b}$ 1

For $x = b$, $1 = B(b-a)$ so $B = \frac{1}{b-a} = \frac{-1}{a-b}$. 1

Then $$\int \frac{1}{(x-a)(x-b)}\,dx = \frac{1}{a-b}\left[\int \frac{1}{x-a}\,dx - \int \frac{1}{x-b}\,dx\right]$$

$$= \frac{1}{a-b}\left[\ln|x-a| - \ln|x-b|\right] + C$$

$$= \frac{1}{a-b}\ln\left|\frac{x-a}{x-b}\right| + C,$$ 2

where in the last step we used the property of logarithms that $\ln(x) - \ln(y) = \ln\left(\frac{x}{y}\right)$.

61. (a) If $\frac{P(x)}{(x-a)(x-b)} = \frac{A}{x-a} + \frac{B}{x-b}$, then

$P(x) = A(x-b) + B(x-a)$, so

$P(a) = A(a-b)$, $P(b) = B(b-a)$

and $A = \frac{P(a)}{a-b}$, $B = \frac{P(b)}{b-a}$.

Now by the product rule $Q'(x) = (x-a) + (x-b)$

so $Q'(a) = 0 + a - b$. this says $A = \dfrac{P(a)}{Q'(a)}$

Similarly $Q'(b) = b - a + 0$, so $B = \dfrac{P(b)}{Q'(b)}$. 2

(b) If $P(x) = 3x - 2$ and $Q(x) = x^2 - 4x - 12$
$= (x-6)(x+2)$

then when $a = 6$, $b = -2$ we get

$$A = \frac{P(a)}{Q'(a)} = \frac{3 \cdot 6 - 2}{2 \cdot 6 - 4} = \frac{16}{8} = 2$$ 2

$$B = \frac{P(b)}{Q'(b)} = \frac{3 \cdot (-2) - 2}{2(-2) - 4} = \frac{-8}{-8} = 1$$

and it's easy to check

$$\frac{3x-2}{(x-6)(x+2)} = \frac{2}{x-6} + \frac{1}{x+2}.$$

62. this generalizes what we did in 61 above

(a) If $\dfrac{P(x)}{Q(x)} = \dfrac{A_1}{x-a_1} + \cdots + \dfrac{A_n}{x-a_n}$ where $Q(x) = (x-a_1)\cdots(x-a_n)$,

then $P(x) = A_1 \underbrace{(x-a_2)\cdots(x-a_n)}_{\left(\text{all factors in } Q \text{ except } x-a_1\right)} + A_2 \underbrace{(x-a_1)(x-a_3)\cdots(x-a_n)}_{\left(\text{all in } Q(x) \text{ except } x-a_2\right)} + \cdots + A_n \underbrace{(x-a_1)\cdots(x-a_{n-1})}_{\left(\text{all in } Q(x) \text{ except } x-a_n\right)}$

so for each $j = 1, \ldots, n$, substituting $x = a_j$, the factor $(x - a_j)$ is contained in all but the term with A_j and

$$P(a_j) = 0 + \cdots + 0 + A_j (a_j - a_1)\cdots(a_j - a_{j-1})(a_j - a_{j+1})\cdots(a_j - a_n) + 0 + \cdots + 0$$

and $$A_j = \frac{P(a_j)}{(a_j-a_1)\cdots(a_j-a_{j-1})(a_j-a_{j+1})\cdots(a_j-a_n)}$$

But then if $Q(x) = (x-a_1)\cdots(x-a_n)$, by the product rule,

$$Q'(x) = \underbrace{(x-a_2)\cdots(x-a_n)}_{\text{all except } x-a_1} + \underbrace{(x-a_1)(x-a_3)\cdots(x-a_n)}_{\text{all except } x-a_2} + \cdots + \underbrace{(x-a_1)\cdots(x-a_{n-1})}_{\text{all except } x-a_n}$$

so $$Q'(a_j) = (a_j-a_1)\cdots(a_j-a_{j-1})(a_j-a_{j+1})\cdots(a_j-a_n)$$

(all the other terms have the factor $x-a_j$, so they all evaluate to 0.) This shows

$$\boxed{A_j = \frac{P(a_j)}{Q'(a_j)}}.$$ 4

(b) with $P(x) = 2x^2-1$ $\quad Q(x) = (x-(-1))(x-2)(x-3)$
$\qquad = x^3-4x^2+x+6$
the partial fractions are

$$\frac{A_1}{x-a_1} + \frac{A_2}{x-a_2} + \frac{A_3}{x-a_3} \qquad (a_1=-1,\ a_2=2,\ a_3=3)$$

and by part (a): $\quad Q'(x) = 3x^2-8x+1$

$$A_1 = \frac{P(-1)}{Q'(-1)} = \frac{1}{12}$$

4

$$A_2 = \frac{P(2)}{Q'(2)} = \frac{-7}{3}$$

$$A_3 = \frac{P(3)}{Q'(3)} = \frac{17}{4}$$

Section 7.6 / 34

Complete the square under the radical:

$$\int \sqrt{x^2+6x}\,dx = \int \sqrt{(x+3)^2-9}\,dx$$

this shows we want a <u>secant</u> substitution:

let $x+3 = 3\sec\theta$, so $dx = 3\sec\theta\tan\theta\,d\theta$

$$\int \sqrt{(x+3)^2-9}\,dx = \int \sqrt{9(\sec^2\theta-1)}\cdot 3\sec\theta\tan\theta\,d\theta$$

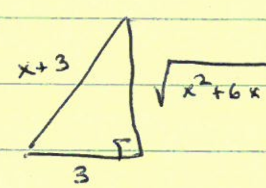
$$= 9\int \sec\theta\tan^2\theta\,d\theta$$

$$= 9\int \sec^3\theta - \sec\theta\,d\theta \quad \text{(trig id)} \quad \tan^2\theta = \sec^2\theta - 1$$

Using the trigonometric reduction formulas, this is

$$9\left[\frac{1}{2}\sec\theta\tan\theta - \frac{1}{2}\ln|\sec\theta+\tan\theta|\right] + C$$

$$= \frac{9}{2}\cdot\frac{(x+3)}{3}\cdot\frac{\sqrt{x^2+6x}}{3} - \frac{9}{2}\ln\left|\frac{x+3}{3}+\frac{\sqrt{x^2+6x}}{3}\right| + C$$



$$= \boxed{\frac{(x+3)\sqrt{x^2+6x}}{2} - \frac{9}{2}\ln\left|\frac{x+3}{3}+\frac{\sqrt{x^2+6x}}{3}\right| + C}$$

$\sec\theta = \frac{x+3}{3}$