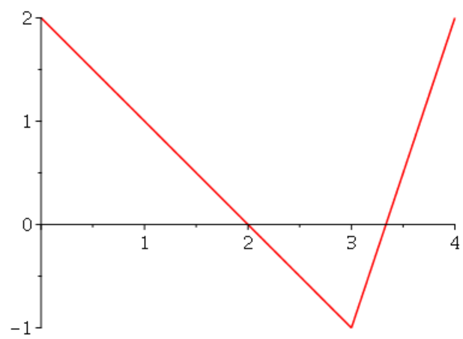
College of the Holy Cross
MATH 134, Calculus With Fundamentals 2
Final Examination Solutions – Thursday, May 10

I. Integration and the Fundamental Theorem of Calculus.

(A) (10) Let $f(x) = \begin{cases} 2 - x & \text{if } 0 \le x \le 3 \\ 3x - 10 & \text{if } 3 < x \le 4. \end{cases}$ whose graph is shown here:



Let $F(x) = \int_0^x f(t)\,dt$. Complete the following table of values for $F(x)$ and $F'(x)$. (Assume the graph continues to the left and the right so $F'(0)$ and $F'(4)$ make sense.)

Solution: For the values of $F(x)$, we compute *signed areas* using the formulas for rectangles and triangles. The second line crosses the x-axis at $x = 10/3$, so the area of the small triangle below the x-axis between $x = 3$ and $x = 4$ is $1/6$ and the area of the larger triangle above the axis is $2/3$. The entry for $F(4) = 3/2 - 1/6 + 2/3 = 2$. The values of $F'(x)$ are the same as $f(x)$ by Part II of the Fundamental Theorem:

$$F'(x) = \frac{d}{dx}\int_0^x f(t)\ dt = f(x).$$

x	0	1	2	3	4
$F(x)$	0	3/2	2	3/2	2
$F'(x)$	2	1	0	−1	2

(B) (10) Compute $G'(x)$ if $G(x) = \displaystyle\int_0^{x^3} e^{-x^2}\ dx$.

Solution: By Part II of the Fundamental Theorem and the chain rule:

$$G'(x) = e^{-(x^3)^2} \cdot 3x^2 = 3x^2 e^{-x^6}.$$

II. Compute the following integrals using basic rules, u-substitution, or integration by parts, as appropriate.

(A) (10) $\int x^2 \cos(x^3)\ dx$

Solution: Let $u = x^3$, then $du = 3x^2\ dx$ so the integral is

$$\frac{1}{3}\int \cos(u)\ du = \frac{1}{3}\sin(u) + C = \frac{1}{3}\sin(x^3) + C$$

(B) (10) $\int x^2 e^{5x}\ dx$

Solution: Integrate by parts twice letting $u = x^2$ the first time and $u = x$ the second (after factoring out constants):

$$\begin{aligned}\int x^2 e^{5x}\ dx &= \frac{x^2}{5}e^{5x} - \frac{2}{5}\int xe^{5x}\ dx \\ &= \frac{x^2}{5}e^{5x} - \frac{2}{5}\left(\frac{x}{5}e^{5x} - \int \frac{1}{5}e^{5x}\right) + C \\ &= \frac{x^2}{5}e^{5x} - \frac{2x}{25}e^{5x} + \frac{2}{125}e^{5x} + C.\end{aligned}$$

(C) (10) $\int x^2(1 - 4x + 3x^2)\ dx$

Solution: Multiply out and integrate term by term using the power rule:

$$= \int x^2 - 4x^3 + 3x^4\ dx = \frac{x^3}{3} - x^4 + \frac{3x^5}{5} + C.$$

III. Trigonometric Substitutions.

A. (10) What trigonometric substitution would you use to evaluate

$$\int \frac{x^2}{\sqrt{49 - x^2}}\ dx?$$

Make the substitution and simplify so there is no square root in the resulting trigonometric integral.

Solution: We use $x = 7\sin(\theta)$, so $dx = 7\cos(\theta)\ d\theta$ and the integral is

$$\begin{aligned}\int \frac{49\sin^2(\theta)}{\sqrt{49 - 49\sin^2(\theta)}} \cdot 7\cos(\theta)\ d\theta &= \int \frac{49\sin^2(\theta)}{\sqrt{49\cos^2(\theta)}} \cdot 7\cos(\theta)\ d\theta \\ &= 49\int \sin^2(\theta)\ d\theta\end{aligned}$$

B. (10) Find $\int \sin^2(\theta)\cos^2(\theta)$ using the trigonometric reduction formulas.

Solution: Using SC3 (with $n = 2$, $m = 2$) then SC2 (with $n = 2$) we get

$$\begin{aligned}\int \sin^2(\theta)\cos^2(\theta) &= \frac{-\sin(\theta)\cos^3(\theta)}{4} + \frac{1}{4}\int \cos^2(\theta)\ d\theta \\ &= \frac{-\sin(\theta)\cos^3(\theta)}{4} + \frac{\cos(\theta)\sin(\theta)}{8} + \frac{\theta}{8} + C.\end{aligned}$$

C. (10) Suppose you had used the trig substitution $x = 3\tan(\theta)$ and you integrated the resulting trigonometric integral to get

$$\theta + \sec(\theta) + C$$

What is the final answer, expressed as a function of x?

Solution: From the triangle with $\tan(\theta) = \frac{x}{3}$, $\sec(\theta) = \dfrac{\text{hypotenuse}}{\text{adjacent}}$, so the answer is

$$\tan^{-1}(x/3) + \frac{\sqrt{x^2+9}}{3} + C.$$

IV. Partial Fractions.

A. (5) How do you recognize the situation where you need to start by using polynomial division? Describe briefly.

Solution: It is when the highest power of the variable appearing in the numerator is greater than or equal to the greatest power appearing in the denominator of the rational function.

B. (10) To decompose $\dfrac{x^2+3}{(x+1)(x+2)^2(x^2+4)}$ into partial fractions, what would the form of the fractions be (leave coefficients as undetermined; *do not try to solve for them*).

Solution: The partial fractions would be

$$\frac{A}{x+1} + \frac{B}{x+2} + \frac{C}{(x+2)^2} + \frac{Dx+E}{x^2+4}.$$

C. (10) Determine the values A, B, C making

$$\frac{3x+1}{x(x+3)(x-1)} = \frac{A}{x} + \frac{B}{x+3} + \frac{C}{x-1}$$

Solution: Clearing the denominators,

$$3x + 1 = A(x+3)(x-1) + Bx(x-1) + Cx(x+3).$$

Setting $x = 0$ we see $1 = -3A$ so $A = -1/3$. Setting $x = -3$, we get $-8 = 12B$ so $B = -2/3$. Finally, setting $x = 1$ gives $4 = 4C$ so $C = 1$.

D. (15) You have decomposed a rational function $f(x)$ into partial fractions as

$$f(x) = 3x^3 + 4 + \frac{6}{x} + \frac{2}{x^2} + \frac{2x-3}{x^2+4}.$$

What is $\int f(x)\, dx$?

Solution: The integral is

$$\frac{3x^4}{4} + 4x + 6\ln|x| - \frac{2}{x} + \ln|x^2+4| - \frac{3}{2}\tan^{-1}(x/2) + C.$$

V. All parts of this problem refer to the region R bounded by $y = x$, $y = 4 - x^2$, $x = 0$ and $x = 1$.

A. (10) Sketch the region R.

Solution: $y = 4 - x^2$ is a parabola opening down; $y = x$ is a straight line with slope $m = 1$ going through the origin. The parabola lies above the line over the whole interval of x values.

B. (10) Compute the area of the region R.

Soltuion: The area is

$$A = \int_0^1 4 - x^2 - x\; dx = 4x - \frac{x^3}{3} - \frac{x^2}{2}\Big|_0^1 = \frac{19}{6}.$$

C. (10) Compute the volume of the solid of revolution obtained by rotating the region R about the x-axis.

Solution: The solid has washer cross-sections by planes $x =$ constant. The inner radius is $y = x$; the outer radius is $y = 4 - x^2$. So the volume is

$$V = \int_0^1 \pi(4-x^2)^2 - \pi(x)^2\; dx = \pi \int_0^1 x^4 - 9x^2 + 16\; dx = \pi\left(\frac{x^5}{5} - 3x^3 + 16x\right|_0^1 = \frac{66\pi}{5}.$$

VI. Other Applications of Integrals.

(A) (10) Find the general solution of the differential equation $\frac{dy}{dx} = e^y \cdot x$.

Solution: Separate variables and integrate:

$$\begin{aligned} e^{-y}\; dy &= x\; dx \\ \int e^{-y}\; dy &= \int x\; dx \\ -e^{-y} &= \frac{x^2}{2} + C \\ y &= -\ln\left(-\frac{x^2}{2} - C\right) \\ &= \ln\left(\frac{1}{\frac{-x^2}{2} - C}\right). \end{aligned}$$

(B) (10) Suppose that a random variable T has pdf $f(t) = 20t^3(1-t)$ for $0 \le t \le 1$ (and 0 otherwise). Find $P\left(0 \le T \le \frac{1}{3}\right)$.

Solution: The probability is given by

$$\int_0^{1/3} 20t^3 - 20t^4 \; dt = 5t^4 - 4t^5\Big|_0^{1/3} = \frac{5}{81} - \frac{4}{243} = \frac{11}{243}.$$

VII. Miscellany – *answer any 3 of the following.* If you answer more than 3, I will count all points received giving a possibility of up to 20 points Extra Credit.

(A) (10) The standard normal pdf is the function $f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$. Use a left-hand Riemann sum with $N = 5$ to approximate $\int_0^{0.5} f(x)\; dx$.

Solution: We have $\Delta x = .1$. The value is

$$\sum_{j=1}^{5} \frac{1}{\sqrt{2\pi}} e^{-((j-1)\cdot(0.1))^2/2} \cdot (0.1) \doteq .1937.$$

(B) (10) If Z has a standard normal distribution, what is $P(0 \le Z \le 0.50)$ from our table? Why is this number smaller than the answer in part A? Explain briefly.

Solution: The value from the table is .1915. This is smaller because the fuction $f(x)$ is decreasing on the interval $[0, 1]$.

(C) (10) What is the future value of a series of monthly payments of \$300 at 5% interest, over a period of 6 years? If you were making these payments to pay off a car loan for \$18000 would you have overpaid or still owe the loan company?

Solution: The future value of the series of payments is

$$\$300 \cdot \left(\frac{(1 + .05/12)^{6 \cdot 12} - 1}{.05/12} \right) = \$25,129.28$$

The future value of the \$18000 is

$$\$18000 \cdot \left(1 + \frac{.05}{12}\right)^{6 \cdot 12} = \$24,282.32$$

You overpaid by the difference: \$846.96.

(D) (10) According to the *multiplier effect* in economics, when there is an injection of money to consumers in an economy, the consumers spend a certain proportion of it, then that amount recirculates through the enconomy and adds additional income, which comes back to the consumers and they spend the same percentage, etc. The process repeats *indefinitely* circulating additional money through the economy. Suppose the government cuts taxes by \$ 50 billion, thereby giving that much money back to consumers. If consumers save 10% of the money they get and spend the other 90%, what is the total additional spending circulated through the economy by the tax cut?

Solution: Of the initial \$50 billion, the consumers spend $50 \cdot .9 = 45$ billion, which gets recirculated. They spend an additional $50 \cdot (.9)^2$ billion, which gets recirculated etc. If this process continues indefinitely, the total is

$$50 \cdot .9 + 50 \cdot (.9)^2 + 50 \cdot (.9)^3 + \cdots$$

This is the sum of the (infinite) geometric series with first term 45 and ratio .9, so it equals

$$\lim_{N \to \infty} \frac{45 - 45 \cdot (.9)^N}{1 - .9} = \frac{45}{1 - .9} = 450$$

(billion dollars).

(E) (10) Let $y(t)$ represent the population of a colony of tree frogs that is undergoing logistic growth following the differential equation $\frac{dy}{dt} = (.1)y(1 - \frac{y}{100})$, t in years. If the initial population is $y(0) = 10$, how long does it take for the population to reach 45?

Solution: We know the value of $k = .1$ from the equation so the solution has the form

$$y(t) = \frac{100}{1 + ce^{-.1t}}$$

With $t = 0$, we get $10 = \frac{100}{1+c}$ so $c = 9$. Then we need to solve

$$45 = \frac{100}{1 + 9e^{-.1t}}$$

for t. The answer is $t \doteq 19.97$, or almost 20 years.