Mathematics 134 – Calculus 2 With Fundamentals
Exam 4 – Solutions for Review Sheet Sample Problems
April 26, 2018

Sample problems

Note: The actual exam will be considerably shorter than the following list of questions and it might not contain questions of all of these types. The purpose here is just to give an idea of the range of different topics that will be covered and how questions might be posed.

I. For each of the following integrals, say why the integral is improper, determine if the integral converges, and if so, find its value.

A) $\displaystyle\int_1^\infty \frac{1}{\sqrt[5]{x}}\,dx$

Solution: This is improper because of the infinite interval.

$$\lim_{b\to\infty}\int_1^b x^{-1/5}\,dx = \lim_{b\to\infty}\frac{5}{4}x^{4/5}\bigg|_1^b = \lim_{b\to\infty}\frac{5}{4}\left(b^{4/5}-1\right).$$

This is not finite, so the integral *diverges – it does not converge.*

B) $\displaystyle\int_0^2 \frac{dx}{x^2-7x+6}$

Solution: This one is improper because the denominator $x^2-7x+6=(x-1)(x-6)$ is zero at $x=1$, which is in the interior of the interval. In order for the integral to converge, *both*

$$\lim_{b\to 1^-}\int_0^b \frac{dx}{x^2-7x+6}$$

and

$$\lim_{a\to 1^+}\int_a^2 \frac{dx}{x^2-7x+6}$$

must exist. However neither integral does exist. We integrate the first by partial fractions and find

$$\begin{aligned}\lim_{b\to 1^-}\int_0^b \frac{dx}{x^2-7x+6} &= \lim_{b\to 1^-}\left(-\frac{1}{5}\ln|x-1|+\frac{1}{5}\ln|x-6|\bigg|_0^b\right)\\ &= \lim_{b\to 1^-}\left(-\frac{1}{5}\ln|b-1|+\frac{1}{5}(\ln|b-6|-\ln(6))\right).\end{aligned}$$

But the first term here does not have a finite limit as $b\to 1^-$. Therefore this integral also *diverges.*

C) $\int_0^\infty xe^{-3x}\,dx$

Solution: We integrate by parts with $u = x$, $dv = e^{-3x}\,dx$ and find

$$\begin{aligned} &= \lim_{b\to\infty} -\frac{xe^{-3x}}{3} - \frac{e^{-3x}}{9}\Big|_0^b \\ &= \lim_{b\to\infty} -\frac{b}{3e^{3b}} - \frac{1}{9e^{3b}} + \frac{1}{9} \\ &= \frac{1}{9} \end{aligned}$$

using L'Hopital's Rule on the first term (the limit of each of the first two terms is zero). This integral *converges*

D) For which values of a is $\int_0^\infty e^{ax}\sin(x)\,dx$ convergent? Evaluate the integral for those a.

Solution: The only chance here is if $a < 0$ so the exponential is decaying as $x \to +\infty$. If this is true then we integrate by parts twice to get

$$\int e^{ax}\sin(x)\,dx = \frac{e^{ax}}{1+a^2}(a\sin(x) - \cos(x))$$

since $a < 0$, the exponential goes to zero as $x \to \infty$ and the value of the integral is just the negative of the value at $x = 0$:

$$\int_0^\infty e^{ax}\sin(x)\,dx = \frac{1}{1+a^2}.$$

II. The time t (in minutes) between two successive incoming calls at a phone center is a random variable with a pdf of the form $f(t) = ce^{-t/r}$ if $t \geq 0$ and 0 otherwise.

(A) What must the value of c be to make this a valid pdf? Express c in terms of the constant r.

Solution: We want

$$\int_0^\infty ce^{-t/r}\,dt = 1,$$

so

$$1 = \lim_{b\to\infty}\int_0^b e^{-t/r}\,dt = c\lim_{b\to\infty} -re^{-t/r}\Big|_0^b = c\lim_{b\to\infty} r$$

Hence $c = \frac{1}{r}$.

(B) Suppose that the probability that two successive calls are more than one minute apart is .3. What is the value of r?

Solution: We have

$$.3 = \int_1^\infty \frac{1}{r}e^{-t/r}\,dt = e^{-1/r}$$

So then $-1/r = \ln(.3)$, and $r = -1/\ln(.3) \doteq .8306$.

(C) Using the value of r you determined in part A, find the probability that the two successive calls are at least 5 minutes apart.

Solution: This probability is

$$P(T > 5) \doteq \int_5^\infty \frac{1}{.8306} e^{-t/.8306}\, dt \doteq .0024$$

(that is about a .2% chance of this happening).

III. (Normal probabilities) Use the normal curve area table:

(A) The probability that a standard normal random variable Z satisfies $-1.30 < Z < 0.83$

Solution: This is equal to $P(0 < Z < 1.30) + P(0 < Z < 0.83) = .4032 + .2967 = .7999$.

(B) Cans of tomato soup have a stated volume of 16 fluid ounces, but the actual volume of the contents is normally distributed with mean 16.3 fluid oz. and SD .22 fluid oz. What is the probability that a randomly selected can contains less than 16 fluid oz. of tomato soup?

Solution: By the standardization formula, this is equal to

$$P\left(Z < \frac{16 - 16.3}{.22}\right) = P(Z < -1.36) = .5000 - P(0 < Z < 1.36) = .5000 - .4131 = .0869$$

(It would happen about 9% of the time.)

IV. *Differential Equations.*

(A) Verify that for any constant C, the function $y = \sqrt{C + x^2}$ is a solution of the differential equation $\dfrac{dy}{dx} = \dfrac{x}{y}$. Which of these solutions also satisfies the initial condition $y(0) = 5$?

Solution: Computing the derivative with the Chain Rule:

$$y\frac{dy}{dx} = \sqrt{C + x^2} \cdot \frac{2x}{2\sqrt{C + x^2}} = x$$

Therefore $y = \sqrt{C + x^2}$ is a solution.

(B) For which real a is $y = e^{ax}$ a solution of the differential equation $\frac{dy}{dx} - 16y = 0$?

Solution: if $y = e^{ax}$, then $\frac{dy}{dx} = ae^{ax}$. So $\frac{dy}{dx} - 16y = 0$ if and only if $a = 16$.

(C) Find the general solution of

$$\frac{dy}{dx} = xe^{x+y}$$

Solution: This is a separable equation because the right side can be written as

$$\frac{dy}{dx} = xe^x \cdot e^y$$

Hence

$$\int e^{-y}\, dy = \int xe^x\, dx$$

Integrating by parts on the right side,

$$-e^{-y} = xe^x - e^x + C$$

for some arbitrary constant. Hence

$$e^{-y} = -xe^x + e^x - C$$

and

$$y = -\ln(-xe^x + e^x - C).$$

(D) Find the general solution of

$$\sqrt{1-x^2}\frac{dy}{dx} = y^2 + 2y$$

Solution: This is a separable equation:

$$\frac{dy}{y(y+2)} = \frac{1}{\sqrt{1-x^2}}\, dx$$

Integrate with partial fractions on the left:

$$\frac{1}{y(y+2)} = \frac{A}{y} + \frac{B}{y+2}$$

when $A = \frac{1}{2}$ and $B = -\frac{1}{2}$. So

$$\begin{aligned} \int \frac{1/2}{y} - \frac{1/2}{y+2}\, dy &= \int \frac{1}{\sqrt{1-x^2}}\, dx \\ \frac{1}{2}\ln|y| - \frac{1}{2}\ln|y+2| &= \sin^{-1}(x) + C \\ \ln\left|\frac{y}{y+2}\right| &= 2\sin^{-1}(x) + C' \qquad (C' = 2C) \\ y &= \frac{2ae^{2\sin^{-1}(x)}}{1 - ae^{2\sin^{-1}(x)}}. \end{aligned}$$

where $a = \pm e^{C'}$.

(E) Find the general solution of

$$\frac{dy}{dx} = xe^{x+y}$$

Solution: This is also separable:

$$\int e^{-y}\,dy = \int xe^{x}\,dx.$$

On the right integrate by parts with $u = x$ $dv = e^{x}\,dx$. So

$$-e^{-y} = xe^{x} - e^{x} + C$$

Then

$$y = -\ln(-xe^{x} + e^{x} + C)$$

(I have written the arbitrary constant as C again; in the final version, the constant is the negative of the one in the line before.)

(F) A baked potato is taken out of the oven at a temperature of 140° C and left to cool on a counter in a room maintained at 20° C. After 2 minutes the temperature of the potato has decreased to 100° C. How long will it take for the temperature to reach 80° C?

Solution: The relevant differential equation is the one from Newton's Law of Cooling. The initial condition comes from thinking of the time when the potato comes out of the oven as $t = 0$. Letting T be the temperature of the potato in degrees C, t be time in minutes:

$$\begin{aligned}\frac{dT}{dt} &= k(T-20)\\ T(0) &= 140\end{aligned}$$

The solution is

$$T(t) = 20 + 120e^{kt}$$

(with $k < 0$). We are given $T(2) = 100$. So

$$100 = 20 + 120e^{2k}$$

and $k = \frac{1}{2}\ln(2/3) \doteq -.20273$. Then we want to solve for t in the equation from the time when $T = 80$:

$$80 = 20 + 120e^{(-.20273)t}$$

which shows

$$t = \frac{\ln(1/2)}{-.20273} \doteq 3.42$$

It will take a total of 3.42 minutes (equivalently, another 1.42 minutes after the time when $T = 100$).

(G) Sunset Lake is stocked with 2000 rainbow trout. After 2 years the trout population has grown to 4500. Assuming logistic growth with a carrying capacity of 10000 trout, find the growth constant k and determine when the population will reach 5000.

Solution: Let y be the trout population as a function of time t in years. We are given that the growth is logistic, which means that

$$\frac{dy}{dt} = ky\left(1 - \frac{y}{10000}\right)$$

for some k. We also know the general solution will look like this:

$$y = \frac{10000}{1 + ae^{-kt}}$$

Take $t = 0$ to be the time when the lake is stocked, so $y(0) = 2000$. (We need to assume there were no trout there before.) Then

$$2000 = \frac{10000}{1+a} \Rightarrow a = 4.$$

Next at $t = 2$ we have

$$\begin{aligned} 4500 &= \frac{10000}{1 + 4e^{-2k}} \\ 1 + 4e^{-2k} &= \frac{10000}{4500} \doteq 2.222 \\ k &\doteq -\frac{1}{2}\ln(1.222/4) \doteq .5929 \end{aligned}$$

The population will reach 5000 when

$$5000 = \frac{10000}{1 + 4e^{-(.5929)t}}.$$

Solving for t:

$$t = \frac{\ln(1/4)}{-.5929} \doteq 2.3$$

years. That is, in another 0.3 years after $t = 2$.

V. Time value of money (and geometric series).

(A) Juan deposits \$500 in a savings account earning 3% interest compounded monthly. What will the balance in the account be after 4 years? What if the interest is compounded continuously.

Solution: We want the future value of a single sum of \$500 after 4 years with a 3% annual interest rate, compounded monthly. This is:

$$FV = \$500 \cdot \left(1 + \frac{.03}{12}\right)^{12 \cdot 4} \doteq \$563.66$$

With continuous compounding the balance will be

$$\$500 \cdot e^{(.03)(4)} \doteq \$563.75$$

or 9 cents more(!)

(B) What is the sum of the finite geometric series

$$6 + \frac{6}{4} + \frac{6}{16} + \cdots + \frac{6}{4^9}?$$

What happens when you keep adding terms of this form indefinitely:

$$6 + \frac{6}{4} + \frac{6}{16} + \cdots + \frac{6}{4^{N-1}}$$

and take a limit as $N \to \infty$?

Solution: The given sum is

$$\frac{6\left(1 - \left(\frac{1}{4}\right)^{10}\right)}{1 - \frac{1}{4}} = \frac{1048575}{131072}$$

(Needless to say the expression to the left of the equals sign is more informative!) For the second part of the question, note that $\lim_{N\to\infty} \frac{1}{4^N} = 0$. So

$$\lim_{N\to\infty} \frac{6\left(1 - \left(\frac{1}{4}\right)^{N}\right)}{1 - \frac{1}{4}} = \frac{6}{1 - \frac{1}{4}} = 8.$$

In fact, the decimal equivalent of $\frac{1048575}{131072}$ is 7.999992371(!) so we're already extremely close to 8 just with the first 10 terms in the geometric series.

(C) Some banks began offering "Christmas clubs" around the time of the Great Depression in the 1930's. Customers would make contributions from their weekly pay checks to the "club" in order to have a given amount on hand to buy presents come Christmas time. Suppose you decide to make 48 weekly payments into a Christmas club earning an annual interest rate of 1%, compounded weekly. What should the payment be in order to have \$800 on hand to spend on gifts at the end of the 48 weeks?

Solution: If the weekly payment is P, the future value of the stream of 48 payments is

$$P \cdot \frac{\left(1 + \frac{.01}{52}\right)^{48} - 1}{\frac{.01}{52}} \doteq P \cdot 48.21763960$$

We want this to equal \$800 for the Christmas gifts, so

$$P = \frac{800}{48.21763960} \doteq \$16.59$$

(Note: The bank could make money on the deal by offering a lower interest rate on the Christmas club than they could get by reinvesting the money held for the customers in other ways – e.g. by writing other loans. And the interest-bearing "club" is only a slight improvement over stashing money under the mattress – the total of the "club" payments is

$$48 \cdot \$16.59 = \$796.32$$

The point of these arrangements was to give the customers an easy way to set aside money for a bigger-ticket item that was predictable every year. It was an encouragement to save.)

(D) What would be the monthly payment on a 15-year mortgage for $\$300,000$ at an annual interest rate of 3.6%?

Solution: As we discussed in class, these loans are structured so that the future value of the stream of monthly payments is equal to the future value of the loan amount given to the customer at the start of the loan. The monthly payment amount is

$$q = \frac{P \cdot \frac{r}{12}}{1 - \left(1 + \frac{r}{12}\right)^{-12 \cdot y}}$$

if the loan is written for a term of y years. With the given information

$$q = \frac{\$300000 \cdot (.036/12)}{1 - (1 + .036/12)^{-180}} \doteq \$2159.41$$