College of the Holy Cross, Spring 2018
Math 134 Midterm Exam 4 Solutions
Friday, May 4

I. The time t (in seconds) between two successive incoming calls at a very busy consumer service call center is a random variable with pdf

$$f(t) = \begin{cases} 5e^{-0.2t} & \text{if } t \geq 0 \\ 0 & \text{if } t < 0 \end{cases}$$

Notes:

1. This was a *defective problem.* The given function is *not* a valid pdf because the factor of 5 should be $\frac{1}{5}$ instead. This was entirely my fault and I gave people full credit on B for the correct value of the integral with the 5 rather than the $\frac{1}{5}$. However that was 20.47, which of course is not a possible value for a probability – probabilities have to be between 0 and 1.

2. In addition, only a few people (two or three) made any real progress on part C, so I threw that part out and gave people Extra Credit if they did that correctly or close to correctly given the information you had.

The solutions below use the correct form of the pdf I wanted:

$$f(t) = \begin{cases} \frac{1}{5}e^{-0.2t} & \text{if } t \geq 0 \\ 0 & \text{if } t < 0 \end{cases}$$

A. (5) Circle the integral that represents the probability that the time between successive calls is *at least 1* second:

   Solution: The correct integral is:

$$\int_1^\infty \frac{1}{5}e^{-0.2t}\, dt$$

B. (10) Evaluate the integral you circled in part A.

   Solution: The value is:

$$\begin{aligned}\int_1^\infty \frac{1}{5}e^{-0.2t}\, dt &= \lim_{b\to\infty} \int_1^b \frac{1}{5}e^{-0.2t}\, dt \\ &= \lim_{b\to\infty} -e^{-0.2t}\Big|_1^b \\ &= \lim_{b\to\infty} -e^{-0.2b} + e^{-0.2} \\ &= 0 + .8187\end{aligned}$$

   That is, this happens about 82% of the time.

C. (10) The median time between calls is the time m for which $P(t \le m) = P(t \ge m) = 1/2$. Determine the median time between calls at this center.

Solution: We can determine the median time by setting up the equation

$$0.5 = P(t < m) = \int_0^m \frac{1}{5} e^{-0.2t}\, dt,$$

integrating, then solving for m:

$$\begin{aligned} 0.5 &= \int_0^m \frac{1}{5} e^{-0.2t}\, dt \\ &= -e^{-0.2t}\Big|_0^m \\ &= 1 - e^{-0.2m}. \end{aligned}$$

Hence $e^{-0.2m} = 0.5$, so $m = \frac{\ln(0.5)}{-0.2} \doteq 3.466$ (seconds).

II. Use the attached table of normal curve areas to answer the following questions.

A. (10) If Z has a standard normal distribution, what is $P(-1.00 < Z < 1.34)$?

Solution: By the symmetry of the normal pdf about 0, this is equal to

$$P(0 < Z < 1) + P(0 < Z < 1.34) = .3413 + .4099 = .7512$$

B. (10) Bottles of cream soda have a stated volume of 20 fluid ounces, but the actual volume of the contents is normally distributed with mean 20.2 fluid oz. and SD = .31 fluid oz. What is the probability that a randomly selected bottle contains more than 20 fluid oz. of soda?

Solution: We standardize and use the table. This is

$$P(Z > \frac{20 - 20.2}{.31}) \doteq P(Z > -.65) = P(0 < Z < +.65) + .5000 = .2422 + .5000 = .7422$$

III. (Differential Equations)

A. (20) Find the general solution of the differential equation

$$\frac{dy}{dx} = (1 + y^2)x^3 \ln(x)$$

Solution: We separate variables and integrate like this:

$$\begin{aligned} \frac{1}{1+y^2}\, dy &= x^3 \ln(x)\, dx \\ \int \frac{1}{1+y^2}\, dy &= \int x^3 \ln(x)\, dx \qquad \text{(use parts with } u = \ln(x)) \\ \tan^{-1}(y) &= \frac{x^4}{4}\ln(x) - \frac{x^4}{16} + C \\ y &= \tan\left(\frac{x^4}{4}\ln(x) - \frac{x^4}{16} + C\right) \end{aligned}$$

B. (5) Newton's Law of Heating/Cooling says that if an object is placed into a surrounding medium of constant temperature A, then the rate of change of the object's temperature is proportional to the difference between the object's temperature and the surrounding temperature. Circle the differential equation that corresponds to this statement:

Solution: The correct differential equation is:

$$\frac{dy}{dt} = k(y - A)$$

C. (15) A roasted chicken is taken out of the oven at a temperature of $190°C$ and left to cool on a counter in a room maintained at a constant temperature of 20° C. After 10 minutes the temperature of the chicken has decreased to 100° C. How long will it take for the temperature to reach 45° C?

Solution: The general solution is

$$y = A + ce^{kt} = 20 + ce^{kt}.$$

Taking $t = 0$ to correspond to the time when the chicken comes out of the oven, $y(0) = 190$, so $190 = 20 + c$ and $c = 170$. Then $y = 20 + 170e^{kt}$. At $t = 10$ minutes, the temperature of the chicken is 100, so $100 = 20 + 170e^{10k}$, so $k = \frac{\ln(8/17)}{10} \doteq -0.0754$. Then the chicken reaches 45 when $45 = 20 + 170e^{-0.0754t}$ so

$$t = \frac{\ln(25/170)}{-0.0754} \doteq 25.43$$

(minutes).

IV.

A. (5) What is the future value of $100,000 at an interest rate of 5%, compounded monthly for 15 years?

Solution: This is

$$\$100,000\left(1 + \frac{.05}{12}\right)^{15\cdot 12} \doteq \$211,370.41$$

B. (5) Suppose you make monthly payments of $700 for 15 years at 5%, what is the future value of that stream of payments?

Solution: By the geometric series formula, the future value is

$$\$700 \cdot \frac{\left((1 + \frac{.05}{12})^{15\cdot 12} - 1\right)}{\frac{.05}{12}} \doteq \$187,122.28$$

C. (5) Suppose you were making those payments to a bank in return for a loan of $100,000. At the end of the 15 years, would you still owe the bank, or would you have overpaid?

Solution: You would still owe the bank the difference of the amounts from parts A and B:

$$\$211,370.41 - \$187,122.28 = \$24268.12.$$

(A fully-amortized loan would be set up with a monthly payment greater than \$700 to make the future values from parts A and B *equal*.)