College of the Holy Cross, Spring 2018
Math 134 Midterm Exam 1 Solutions
Friday, February 16

I. Both parts of this problem refer to $f(x) = -x^2 + 4$ on the interval $[a, b] = [0, 2]$.

A. (15) Evaluate the R_4 Riemann sum for f on this interval.

Solution: We have $\Delta x = \frac{2-0}{4} = \frac{1}{2}$, so $x_1 = \frac{1}{2}, x_2 = 1, x_3 = \frac{3}{2}$, and $x_4 = 2$. The right hand sum is

$$\begin{aligned} R_4 &= (-(1/2)^2 + 4)(1/2) + (-(1)^2 + 4)(1/2) + (-(3/2)^2 + 4)(1/2) + (-(2)^2 + 4)(1/2) \\ &= (15/4)(1/2) + (3)(1/2) + (7/4)(1/2) + 0 \\ &= 34/8 = 4.25. \end{aligned}$$

B. (10) Use Part I of the Fundamental Theorem of Calculus to evaluate $\int_0^2 -x^2 + 4\, dx$.

Solution: We have

$$\begin{aligned} \int_0^2 -x^2 + 4\, dx &= \frac{-x^3}{3} + 4x\Big|_0^2 \\ &= \frac{-8}{3} + 8 - 0 \\ &= \frac{16}{3}. \end{aligned}$$

C. (5) Your answer to part B should be larger than your answer to part A. Explain how you could know it would turn out that way even without computing the numerical values.

Solution: We can tell the integral will be less than the right hand sum because the function $-x^2 + 4$ is *decreasing* on $[0, 2]$ (note, for instance, that $f'(x) = -2x < 0$ for all x in $[0, 2]$). This implies that the actual value of the integral is larger than all right-hand sums.

II. (10) The following limit of a sum would equal the definite integral $\int_a^b f(x)\, dx$ for some function $f(x)$ on some interval $[a, b]$. What function and what interval?

$$\lim_{N\to\infty} \sum_{j=1}^{N} \left(2 + \left(\frac{2j}{N}\right)^4\right) \cdot \frac{2}{N}.$$

Solution: The sum is the R_N sum for $f(x) = 2 + x^4$ on $[0, 2]$, so the integral is

$$\int_0^2 2 + x^4\, dx.$$

III. Both parts of this problem refer to $f(x) = (x - 2)^2(x - 4)$.

A. (10) Let $A(x) = \int_0^x f(t)\, dt$. What is $A'(x)$?

Solution: By the FTC Part II,

$$A'(x) = f(x) = (x-2)^2(x-4).$$

B. (10) Now consider $B(x) = \int_1^{x^3} f(t)\, dt$. What is $B'(x)$?

Solution: By the FTC Part II and the Chain Rule,

$$B'(x) = (x^3-2)^2(x^3-4)\cdot 3x^2.$$

IV. Compute the following integrals.

A. (10) $\int \sqrt{x} + \frac{1}{x} - e^x\, dx$

Solution:

$$\int \sqrt{x} + \frac{1}{x} - e^x\, dx = \frac{2}{3}x^{3/2} + \ln|x| - e^x + C$$

B. (10)

Solution:

$$\int \cos(x) + \frac{1}{\sqrt{1-x^2}}\, dx = \sin(x) + \sin^{-1}(x) + C$$

C. (10) Integrate with a suitable u-substitution: $\int_0^1 (2x^4+1)^{6/5}x^3\, dx$

Solution: Let $u(x) = 2x^4+1$. Then $du = 8x^3\, dx$, so $x^3\, dx = \frac{1}{8}\, du$. When $x = 0$, $u(0) = 1$ and when $x = 1$, $u(1) = 3$. In terms of u the integral becomes

$$\frac{1}{8}\cdot\int_{u=1}^{u=3} u^{6/5}\, du = \frac{1}{8}\cdot\frac{5}{11}\, u^{11/5}\Big|_1^3 = \frac{5}{88}(3^{11/5}-1) \doteq .5802$$

D. (10) Integrate with a suitable u-substitution: $\int (\tan(x))^3\sec^2(x)\, dx$.

Solution: Let $u = \tan(x)$. Then $du = \sec^2(x)\, dx$ and the integral becomes

$$\int u^3\, du = \frac{u^4}{4} + C = \frac{(\tan(x))^4}{4} + C.$$