Mathematics 134 – Calculus 2 With Fundamentals
Solutions for Review Problems Exam 3 – Review Sheet
April 5, 2018

I. Compute each of the integrals below the appropriate method or combination of methods. You must show all work for full credit.

A) Use the trigonometric reduction formulas to find

$$\int \sin^3(x)\cos^4(x)\ dx$$

Solution: Use the SC3 formula first to get

$$\int \sin^3(x)\cos^4(x)\ dx = \frac{-\sin^2(x)\cos^5(x)}{7} + \frac{2}{7}\int \sin(x)\cos^4(x)\ dx$$

The integral that is left is $-\int u^4\ du$ for $u = \cos(x)$. So the final answer is

$$-\frac{\sin^2(x)\cos^5(x)}{7} - \frac{2}{35}\cos^5(x) + C.$$

B) Use the trigonometric reduction formulas to find

$$\int \sec^5(x)\ dx$$

Solution: Apply the ST3 formula twice, then the ST4 formula. Answer:

$$\frac{\sec^3(x)\tan(x)}{4} + \frac{3\sec(x)\tan(x)}{8} + \frac{3}{8}\ln|\sec(x)+\tan(x)| + C$$

.

C)

$$\int \sqrt{25-x^2}\ dx$$

Solution: From the $25 - x^2$, we let $x = 5\sin\theta$, so $dx = 5\cos\theta\ d\theta$. The integral becomes

$$25\int \cos^2(\theta)\ d\theta.$$

Then using the SC2 reduction formula

$$= 25\left(\frac{\cos\theta\sin\theta}{2} + \frac{\theta}{2}\right) + C$$

Finally, from the substitution equation and the reference triangle, $\sin\theta = x/5$, $\theta = \sin^{-1}(x/5)$ and $\cos\theta = \frac{\sqrt{25-x^2}}{5}$. The integral is

$$\frac{x\sqrt{25-x^2}}{2} + \frac{25\sin^{-1}(x/5)}{2} + C.$$

D) What changes in part C if the integral is

$$\int \sqrt{25+x^2}\ dx$$

Work this one out too, start to finish.

Solution: The first thing that changes is that the + sign indicates we want the tangent substitution: $x = 5\tan\theta$, so $dx = 5\sec^2\theta\ d\theta$. The integral becomes

$$25\int \sec^3\theta\ d\theta$$

We apply the ST 3 and ST 4 reduction formulas:

$$= \frac{25\sec\theta\tan\theta}{2} + \frac{25\ln|\sec\theta+\tan\theta|}{2} + C.$$

Finally converting back to functions of x, $\tan\theta = x/5$ and $\sec\theta = \sqrt{25+x^2}5$, so the integral

$$= \frac{x\sqrt{x^2+25}}{2} + \frac{25}{2}\ln\left|\frac{\sqrt{x^2+25}}{5} + \frac{x}{5}\right|.$$

E)

$$\int \frac{x^3+2x+1}{x^2+6x-7}\ dx$$

Solution: To apply partial fractions we need to divide first; we find

$$\frac{x^3+2x+1}{=} x - 6 + \frac{45x-41}{x^2+6x-7}$$

We apply the partial fraction decomposition to the last term on the right:

$$\frac{45x-41}{x^2+6x-7} = \frac{A}{x+7} + \frac{B}{x-1}$$

Hence

$$45x - 41 = A(x-1) + B(x+7)$$

Setting $x = -7$, we get $A = \frac{356}{8} = \frac{89}{2}$. Similarly $x = 1$ gives $4 = 8B$, so $B = \frac{1}{2}$. The integral is

$$\int x - 6 + \frac{315/8}{x+7} + \frac{1/2}{x-1}\ dx = \frac{x^2}{2} - 6x + \frac{89}{2}\ln|x+7| + \frac{1}{2}\ln|x-1| + C.$$

F)

$$\int \frac{1}{x^2(x^2+6x+10)}\ dx$$

Solution: No division necessary here; we go immediately to the partial fractions:

$$\frac{1}{x^2(x^2+6x+10)} = \frac{A}{x} + \frac{B}{x^2} + \frac{Cx+D}{x^2+6x+10}.$$

Clearing denominators:

$$\begin{aligned} 1 &= Ax(x^2+6x+10) + B(x^2+6x+10) + (Cx+D)(x^2) \\ &= (A+C)x^3 + (6A+B+D)x^2 + (10A+6B)x + 10B \end{aligned}$$

Matching coefficients shows $10B = 1$, so $B = \frac{1}{10}$. Then from the coefficient of x, $10A + \frac{6}{10} = 0$, so $A = \frac{-6}{100} = \frac{-3}{50}$. From the coefficient of x^3, $A + C = 0$, so $C = \frac{3}{50}$. Finally, from the coefficient of x^2, $\frac{-18}{50} + \frac{5}{50} + D = 0$. So $D = \frac{13}{50}$ This gives

$$\frac{1}{x^2(x^2+6x+10)} = \frac{-3/50}{x} + \frac{1/10}{x^2} + \frac{(3/50)x + (13/50)}{x^2+6x+10}$$

We have to be slightly tricky to integrate the last term: Complete the square on the bottom: $x^2 + 6x + 10 = (x+3)^2 + 1$. We let $u = x + 3$, $du = dx$, and then $x = u - 3$. So we have

$$\frac{3}{50}\frac{u}{u^2+1} + \frac{4}{50}\frac{1}{u^2+1}$$

The first term here gives a logarithm, the second one is an inverse tangent. The final answer is

$$\frac{-3}{50}\ln|x| - \frac{1}{10x} + \frac{3}{100}\ln|(x+3)^2+1| + \frac{2}{25}\tan^{-1}(x+3) + C.$$

(Comment: This would be a very challenging exam problem because of the trickiness of the arithmetic of the coefficients and the maneuvers required with completing the square in $x^2 + 6x + 10$. It's good practice to work out a few this complicated, though, even if you are not going to see one on the exam ;)

II. (Improper integrals)

A) Why is the following integral an *improper integral*. Decide whether it converges by setting up and evaluating the appropriate limit(s):

$$\int_1^3 \frac{1}{(x-3)^{2/3}}\,dx$$

Solution: This is improper because the function has an infinite discontinuity at $b = 3$ at the right endpoint of the interval of integration. We need to determine

$$\begin{aligned} \lim_{b\to 3^-}\int_1^b (x-3)^{-2/3}\,dx &= \lim_{b\to 3^-} 3(x-3)^{1/3}\Big|_1^b \\ &= \lim_{b\to 3^-} 3(b-3)^{1/3} + 3\cdot 2^{1/3} \\ &= 0 + 3\cdot 2^{1/3}. \end{aligned}$$

Since the limit exists, the improper integral *converges*.

B) Follow-up to part A: For which exponents $\alpha > 0$ will the improper

$$\int_1^3 \frac{1}{(x-3)^{2/3}}\,dx$$

Solution: Thinking about what happened here, the limit of the integral will exist as long as the power $1+\alpha$ in the antiderivative $\frac{1}{1+\alpha}(x-3)^{1+\alpha}$ is *strictly positive*. This means $0 < \alpha < 1$. The exponent $\alpha = 1$ does not "work" since then the integral contains $\ln|x-3|$ which has no limit as $x \to 3^-$.

C) Why is the following integral an improper integral? Decide whether it converges by setting up and evaluating the appropriate limit(s):

$$\int_0^\infty xe^{-x}\,dx.$$

Solution: It's improper because of the infinite upper limit of integration. To decide whether this one converges, we look at

$$\begin{aligned}\lim_{b\to\infty}\int_0^b xe^{-x}\,dx &= \lim_{b\to\infty} -xe^{-x} - e^{-x}\Big|_0^b \quad (\textit{integrate by parts})\\ &= \lim_{b\to\infty}\frac{-b}{e^b} - \frac{1}{e^b} + 1\\ &= 0 + 0 + 1.\end{aligned}$$

(the first term goes to zero as $b \to \infty$ by L'Hopital's Rule). This integral also converges.

III.

(A) Verify that $\int \csc\theta\,d\theta = \ln|\csc\theta - \cot\theta| + C$ by differentiating.

Solution: By the Chain Rule and the derivative formulas for csc and cot,

$$\begin{aligned}\frac{d}{dx}\ln|\csc\theta - \cot\theta| &= \frac{1}{\csc\theta - \cot\theta}\cdot(-\csc\theta\cot\theta + \csc^2\theta)\\ &= \frac{(\csc\theta - \cot\theta)\cdot\csc\theta}{\csc\theta - \cot\theta}\\ &= \csc\theta.\end{aligned}$$

(B) Which trigonometric substitution would you apply to compute $\int \frac{1}{u\sqrt{a^2-u^2}}\,du$? What trigonometric integral do you get after making the substitution? Complete the computation of the integral.

Solution: This calls for the substitution $u = a\sin\theta$, so $du = a\sin\theta\,d\theta$. The integral becomes

$$\int \frac{a\cos\theta\,d\theta}{a\sin\theta\cdot a\cos\theta} = \frac{1}{a}\int \csc\theta\,d\theta.$$

Applying the formula from part (A):

$$= \frac{1}{a} \ln |\csc\theta - \cot\theta| + C.$$

The reference triangle from the substitution has $\sin\theta = u/a$, so we put u on the opposite side and a on the hypotenuse. The adjacent side is $\sqrt{a^2 - u^2}$, so

$$\csc\theta = \frac{a}{u} \qquad \text{and} \qquad \cot\theta = \frac{\sqrt{a^2 - u^2}}{u}.$$

The integral is

$$\frac{1}{a} \ln \left| \frac{a}{u} - \frac{\sqrt{a^2 - u^2}}{u} \right| + C$$

(C) Our textbook's table of integrals gives this one as

$$\frac{-1}{a} \ln \left| \frac{a + \sqrt{a^2 - u^2}}{u} \right|$$

Show that the form you got in part B is equivalent to this.

Solution: By properties of logarithms:

$$\frac{-1}{a} \ln \left| \frac{a + \sqrt{a^2 - u^2}}{u} \right| = \frac{1}{a} \ln \left| \frac{u}{a + \sqrt{a^2 - u^2}} \right|$$

Inside the logarithm, multiply top and bottom of the fraction by the conjugate radical $a - \sqrt{a^2 - u^2}$. On the bottom, expand to $(a^2 - (a^2 - u^2)) = u^2$. Then we can cancel one power of u to yield

$$\frac{1}{a} \ln \left| \frac{a - \sqrt{a^2 - u^2}}{u} \right|$$

Separating the inside into two fractions with denominator u gives the form we derived in part (B).