Mathematics 134 – Calculus 2 With Fundamentals
Exam 2 – Answers/Solutions for Sample Questions
March 2, 2018

*Sample Exam Questions*

*Disclaimer: The actual exam questions may be organized differently and ask questions in different ways. This list is also quite a bit longer than the actual exam will be (to give you some idea of the range of different questions that might be asked).*

I. Compute each of the integrals below using some combination of basic rules, substitution, integration by parts. You must show all work for full credit.

A)

$$\int \frac{e^{\tan^{-1}(x)}}{1+x^2}\ dx$$

*Solution:* By substitution. Let $u = \tan^{-1}(x)$. The form is $\int e^u\ du$, so the integral is $e^{\tan^{-1}(x)} + C$.

B)

$$\int e^x \sin(2x)\ dx$$

*Answer:* Apply integration by parts twice; the second time, the original integral is returned, but with a numerical coefficient, so you can solve for it algebraically. The integral equals: $\frac{1}{5}e^x \sin(2x) - \frac{2}{5}e^x \cos(2x) + C$.

C)

$$\int \sin^3(x)\cos(x)\ dx$$

*Solution:* By substitution. Let $u = \sin(x)$, so $du = \cos(x)\ dx$ and the form is

$$\int u^3\ du = \frac{1}{4}u^4 + C = \frac{1}{4}\sin^4(x) + C.$$

D)

$$\int_0^{1/2} \sin^{-1}(x)\ dx$$

*Answer:* Use integration by parts with $u = \sin^{-1}(x)$ and $dv = dx$. The answer is

$$x\sin^{-1}(x)\Big|_0^{1/2} - \int_0^{1/2} \frac{x}{\sqrt{1-x^2}}\ dx = \frac{\pi}{12} + \sqrt{1-x^2}\Big|_0^{1/2}$$
$$= \frac{\pi}{12} + \frac{\sqrt{3}}{2} - 1.$$

E) Use integration by parts to show this reduction formula: If $n$ is a positive integer, then

$$\int x^n e^{ax}\, dx = \frac{x^n e^{ax}}{a} - \frac{n}{a}\int x^{n-1}e^{ax}\, dx$$

*Solution:* This comes directly from the integration by parts formula if you let $u = x^n$ and $dv = e^{ax}\, dx$, so $du = nx^{n-1}$ and $v = \frac{1}{a}e^{ax}$.

F) Apply the result from part (E) (repeatedly) to compute $\int x^4 e^{-2x}\, dx$.

*Answer:* $\left(\frac{-x^4}{2} - x^3 - \frac{3x^2}{2} - \frac{3x}{2} - \frac{3}{4}\right)e^{-2x} + C$.

G)

$$\int x^8 \ln(x)\, dx$$

*Solution:* Recall that this is the case where we *don't* let $u$ be the power of $x$ in parts. Instead, let $u = \ln(x)$ and $dv = x^8\, dx$. Then $du = \frac{1}{x}\, dx$ and $v = \frac{x^9}{9}$. Then by the parts formula,

$$\begin{aligned}\int x^8 \ln(x)\, dx &= \frac{x^9 \ln(x)}{9} - \int \frac{x^9}{9}\cdot\frac{1}{x}\, dx \\ &= \frac{x^9 \ln(x)}{9} - \int \frac{x^8}{9} dx \\ &= \frac{x^9 \ln(x)}{9} - \frac{x^9}{81} + C.\end{aligned}$$

III.

(A) Let $R$ be the region in the plane bounded by $y = 3 - x^2$ and the $x$-axis.

(1) Sketch the region $R$.

Omitted – the region is extends from $x = -\sqrt{3}$ to $x = +\sqrt{3}$ under the parabola $y = 3 - x^2$, which opens down from the vertex at $(0, 3)$.

(2) Find the area of $R$.

*Answer:* $\int_{-\sqrt{3}}^{\sqrt{3}} 3 - x^2\, dx = 4\sqrt{3}$.

(3) Find the volume of the solid generated by rotating $R$ about the $x$-axis.

*Answer:* $V = \int_{-\sqrt{3}}^{\sqrt{3}} \pi(3 - x^2)^2\, dx = \frac{48\pi\sqrt{3}}{5}$.

(4) Find the volume of the solid generated by rotating $R$ about the line $y = 5$.

*Solution:* The cross-sections are "washers" with inner radius

$$r_{in} = 5 - (3 - x^2) = 2 + x^2$$

and outer radius 5. So the volume is

$$\begin{aligned}\int_{-\sqrt{3}}^{\sqrt{3}} \pi(5)^2 - \pi(2+x^2)^2\ dx &= \pi \int_{-\sqrt{3}}^{\sqrt{3}} 21 - 4x^2 - x^4\ dx \\ &= \pi \left( 21x - \frac{4x^3}{3} - \frac{x^5}{5} \Big|_{-\sqrt{3}}^{\sqrt{3}} \right) \\ &= \frac{152\sqrt{3}\pi}{5}.\end{aligned}$$

(5) Find the volume of the solid generated by rotating $R$ about the line $y = -2$.

*Solution:* The cross-sections are "washers" again, but with inner radius $r_{in} = 0 - (-2) = 2$ and outer radius

$$3 - x^2 - (-2) = 5 - x^2.$$

So the volume is

$$\begin{aligned}\int_{-\sqrt{3}}^{\sqrt{3}} \pi(5-x^2)^2 - \pi(2)^2\ dx &= \pi \int_{-\sqrt{3}}^{\sqrt{3}} 21 - 10x^2 + x^4\ dx \\ &= \pi \left( 21x - \frac{10x^3}{3} + \frac{x^5}{5} \Big|_{-\sqrt{3}}^{\sqrt{3}} \right) \\ &= \frac{128\sqrt{3}\pi}{5}.\end{aligned}$$

(6) Find the volume of the solid generated by rotating $R$ about the line $x = -4$. (Note: This can be done "by slices" or by "shells." Try it both ways and check that you get the same result.

*Solution 1:* By "slices:" The cross-sections by planes perpendicular to the vertical line $x = -4$ are washers. The inner and outer radii come from the difference between $x$ as a function of $y$ on the left and right halves of the parabola: $x = \pm\sqrt{3-y}$ and $x = -4$.

$$r_{in} = -\sqrt{3-y} - (-4) = 4 - \sqrt{3-y}$$

and

$$r_{out} = \sqrt{3-y} - (-4) = 4 + \sqrt{3-y}$$

Hence the volume is computed by

$$\begin{aligned}V &= \int_0^3 \pi(4+\sqrt{3-y})^2 - \pi(4-\sqrt{3-y})^2\ dy \\ &= 16\pi \int_0^3 \sqrt{3-y}\ dy \quad \text{(the other terms cancel)} \\ &= 16\pi \cdot \frac{-2}{3}(3-y)^{3/2} \Big|_0^3 \\ &= 32\sqrt{3}\pi.\end{aligned}$$

*Solution 2:* By "shells:" The volume this way is given by

$$\begin{aligned} V &= \int_{x=-\sqrt{3}}^{x=\sqrt{3}} 2\pi(x-(-4))(3-x^2)\ dx \\ &= 2\pi \int_{-\sqrt{3}}^{\sqrt{3}} 12+3x-4x^2-x^3\ dx \\ &= 2\pi\ 12x+\frac{3x^2}{2}-\frac{4x^3}{3}-\frac{x^4}{4}\Big|_{-\sqrt{3}}^{\sqrt{3}} \\ &= 2\pi(24\sqrt{3}-8\sqrt{3}) \\ &= 32\sqrt{3}\pi. \end{aligned}$$

(The two methods give the same volume, of course!)

(B) Let $R$ be the region in the plane bounded by $y=3x$ and $y=x^2$.

(1) Sketch the region $R$.

Omitted. The curves cross at $(0,0)$ and $(3,9)$.

(2) Find the area of $R$.

*Answer:* $\int_0^3 3x-x^2\ dx=\frac{9}{2}$.

(3) Find the volume of the solid generated by rotating $R$ about the $x$-axis.

*Answer:* The cross sections are washers with outer radius $3x$ and inner radius $x^2$ so $V=\int_0^3 \pi(3x)^2-\pi(x^2)^2\ dx=\frac{162\pi}{5}$.

(4) Find the volume of the solid generated by rotating $R$ about the $y$-axis.

*Answer:* The cross-sections are washers again, but the outer radius at $y$ is $x=\sqrt{y}$ and the inner radius is $x=y/3$. The volume is $V=\int_0^9 \pi(\sqrt{y})^2-\pi(y/3)^2\ dy=\frac{27\pi}{2}$.

(C) Let $R$ be the region in the plane bounded by $y=\cos(\pi x)$, $y=1/2$, $x=-1/3$ and $x=1/3$.

(1) Sketch the region $R$.

*Answer:* Graph omitted. Since $\cos(\pi/3)=\cos(-\pi/3)=1/2$, this is a lens-shaped area thickest at $x=0$.

(2) Find the area of $R$.

*Answer:* $A=\int_{-1/3}^{1/3}\cos(\pi x)-\frac{1}{2}\ dx=\frac{\sqrt{3}}{\pi}-\frac{1}{3}$.

(3) Find the volume of the solid generated by rotating $R$ about the $x$-axis.

(a) *Solution:* The cross-sections are washers with inner radius $1/2$ and outer radius $\cos(\pi x)$. The volume is

$$V=\int_{-1/3}^{1/3}\pi\cos^2(\pi x)-\pi(1/2)^2\ dx$$

Letting $u = \pi x$, we see we have to integrate $\int \cos^2 u \; du = \frac{u}{2} + \frac{1}{4}\sin(2u)$ by the double angle formula. So the volume is

$$V = \frac{\pi x}{2} + \frac{1}{4}\sin(2\pi x) - \frac{\pi x}{4}\Big|_{-1/3}^{1/3} = \frac{\sqrt{3}}{4} + \frac{\pi}{6}.$$

(4) Find the volume of the solid generated by rotating $R$ about the line $y = -1$.

*Answer:*

$$V = \int_{-1/3}^{1/3} \pi(1+\cos(\pi x))^2 - \pi(3/2)^2 \; dx = \frac{9\sqrt{3}}{4} - \frac{\pi}{2}.$$

III. The height of a monument is 20m. The horizontal cross-section of the monument at $x$ meters from the top is an isosceles right triangle with legs $x/4$ meters. Is the given information enough to find the volume of the monument? If so, find the volume. If not, say why not.

*Solution:* By Cavalieri's Principle, this *is enough*. In particular, it doesn't actually matter how the cross-sections are stacked on top of each other. The volume is

$$V = \int_0^{20} \frac{1}{2}\left(\frac{x}{4}\right)^2 \; dx = \frac{250}{3}.$$