College of the Holy Cross, Fall Semester, 2017
MATH 133, Midterm 2 Solutions
Friday, October 20

1. An object moves along a straight line path with position given by $x(t) = 4t^2 + 6t$, (t in seconds, x in feet).

   (a) What is the average velocity of the object on the interval $[0, 3]$?

   Solution: The average velocity is

   $$v_{ave} = \frac{x(3) - x(0)}{3 - 0} = \frac{4 \cdot 3^2 + 6 \cdot 3 - 0}{3 - 0} = 18$$

   (units are feet per second).

   (b) Fill in the following table with average velocities computed over the indicated intervals. Using this information, estimate the *instantaneous velocity* at $t = 0$.

   Solution:

interval	$[0, 1]$	$[0, 0.1]$	$[0, 0.01]$	$[0, 0.001]$
ave.vel.	10	6.4	6.04	6.004

   Estimated instantaneous velocity = 6.

2. Answer all parts of this question by referring to the graph $y = f(x)$ in Figure 1.

   (a) (5) $\lim_{x\to 1^-} f(x) = 2$ and $\lim_{x\to 1^+} f(x) = 2$.

   (b) (5) $f(x)$ has a jump discontinuity at $x = 3$.

   (c) (5) True/False: The limit $\lim_{x\to 2} f(x)$ does not exist. False. If so, what is the limit? If not, say why not: Limit is apparently -1.

   (d) (5) True/False: $f(x)$ has an infinite discontinuity in this part of the graph. False. $x =$ ____________. (If not leave this space blank.)

   (e) (5) True/False: $f(x)$ has a removable discontinuity shown in this part of the graph. True. If so, where is it? $x = 1$.

3. Compute *any four* of the following limits. (Only the best four will be counted for your total score.)

   (a) (10)

   $$\lim_{x\to 1} \frac{x^2 + 7x - 8}{x^2 - 1}$$

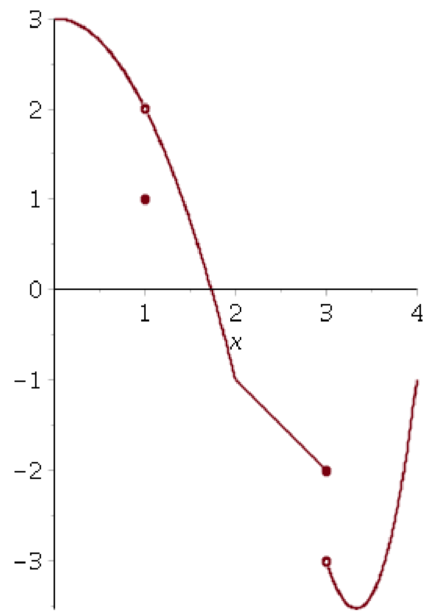

Figure 1: $y = f(x)$ for problem 2.

Solution: This is a 0/0 limit. We can factor, cancel, and evaluate as follows:

$$\begin{aligned} \lim_{x\to 1} \frac{x^2+7x-8}{x^2-1} &= \lim_{x\to 1} \frac{(x-1)(x+8)}{(x-1)(x+1)} \\ &= \lim_{x\to 1} \frac{(x+8)}{(x+1)} \\ &= 9/2. \end{aligned}$$

(b) (10)

$$\lim_{h\to 3} \frac{\sqrt{11+h}-\sqrt{14}}{h-3}$$

Solution: Also a 0/0 indeterminate form. We multiply top and bottom by the conjugate radical, simplify, and then substitute:

$$\begin{aligned}
\lim_{h\to 3}\frac{\sqrt{11+h}-\sqrt{14}}{h-3} &= \lim_{h\to 3}\frac{(\sqrt{11+h}-\sqrt{14})}{h-3}\cdot\frac{(\sqrt{11+h}+\sqrt{14})}{(\sqrt{11+h}+\sqrt{14})}\\
&= \lim_{h\to 3}\frac{11+h-14}{(h-3)(\sqrt{11+h}+\sqrt{14})}\\
&= \lim_{h\to 3}\frac{h-3}{(h-3)(\sqrt{11+h}+\sqrt{14})}\\
&= \lim_{h\to 3}\frac{1}{(\sqrt{11+h}+\sqrt{14})}\\
&= \frac{1}{2\sqrt{14}}.
\end{aligned}$$

(c) (10)

$$\lim_{t\to 0}\frac{t^2+5}{t+1}.$$

Solution: This one is not indeterminate – it can be evaluated directly using the limit quotient, sum, and product rules

$$\lim_{t\to 0}\frac{t^2+5}{t+1}=\frac{0^2+5}{0+1}=5.$$

(d) (10)

$$\lim_{x\to\infty}\frac{x^3+3x+1}{7x^3+x^2+4x}.$$

Solution: We can multiply the top and bottom by $\frac{1}{x^3}$ to obtain:

$$\begin{aligned}
\lim x\to\infty\frac{(x^3+3x+1)\cdot\frac{1}{x^3}}{(7x^3+x^2+4x)\cdot\frac{1}{x^3}} &= \lim_{x\to\infty}\frac{1+\frac{3}{x^2}+\frac{1}{x^3}}{7+\frac{1}{x}+\frac{4}{x^2}}\\
&= \frac{1}{7}.
\end{aligned}$$

(e) (10)

$$\lim_{t\to 0}\frac{\sin(9t)}{t}.$$

Solution: For this one, we want to apply the formula

$$\lim_{x\to 0}\frac{\sin(x)}{x}=1$$

but that requires having exactly the same thing "inside" the sine function and in the denominator. If we multiply top and bottom by 9 we get

$$\lim_{t\to 0}\frac{\sin(9t)}{t}=\lim_{t\to 0}\frac{9\sin(9t)}{9t}=9\cdot 1=9.$$

4. Let $f(x) = x^2 - 2x + 2$.

(a) (5) What is the slope of the secant line to the graph through the points $(1, 1)$ and $(3, 5)$?

Solution: The slope is

$$m_{sec} = \frac{5-1}{3-1} = 2.$$

(b) (5) Give a general formula for the slope of the secant line through the points $(1, 1)$ and $(1+h, (1+h)^2 - 2(1+h) + 2)$.

Solution: The slope of the secant line is

$$\frac{(1+h)^2 - 2(1+h) + 2 - 1}{(1+h) - 1} = \frac{(1 + 2h + h^2) - 2 - 2h + 2 - 1}{h} = \frac{h^2}{h} = h$$

(c) (5) Find the limit as $h \to 0$ of your slope from part (b).

Solution: The limit is 0.

(d) (5) What does your answer in part (c) tell you in terms of the graph $y = f(x)$, related lines, etc.?

Solution: It is the slope of the tangent line to the graph $y = x^2 - 2x + 2$ at $(1, 1)$. The slope is 0 which means the tangent line is the horizontal line $y = 1$.