College of the Holy Cross, Fall Semester, 2017
MATH 133, Midterm 1 Solutions
Friday, September 22

1. (a) (10) Express the set of all x satisfying $|3x - 9| \leq 6$ as an interval or union of intervals.

   Solution: Algebraically, this inequality says $-6 \leq 3x - 9 \leq 6$, so $3 \leq 3x \leq 15$, so $1 \leq x \leq 5$. As an interval this is $[1, 5]$. Geometrically, we could get the same result by dividing by 3 to get $|x - 3| \leq 2$. The numbers at distance at most 2 along the number line from 3 are exactly $x \in [1, 5]$ as before.

   (b) (10) What is the domain of the function $f(x) = \dfrac{\sqrt{3 - x}}{x}$? Any correct form is OK.

   Solution: We must have $3 - x \geq 0$, or $x \leq 3$ for the square root to be defined. We must also have $x \neq 0$ since we cannot divide by zero. These conditions define $(-\infty, 0) \cup (0, 3]$. Another correct way to say this: all real $x \leq 3$, except $x = 0$.

2. (20) The graph $y = f(x)$ and four graphs obtained by transforming it are shown in Figures 1 and 2. Match the given formulas with the corresponding graph.
   (a) $y = f(\frac{1}{2}x)$:$\underline{B}$ (horizonal stretching) (b) $y = \frac{1}{2}f(x)$:$\underline{D}$ (vertical compression) (c) $y = f(2x)$:$\underline{C}$ (horizontal compression)
   (d) Note that there is an extra graph that does not match any of the formulas. A is the Graph that does not match any formula. (It is actually $y = 2f(x)$ (vertical stretching)).

3. (a) (15) Complete the square: $q(x) = 4x^2 - 16x + 24$.

   Solution: By the usual process,

   $$q(x) = 4(x^2 - 4x + 6) = 4((x - 2)^2 + 2) = 8 + 4(x - 2)^2.$$

   (b) (5) What is the minimum value of $q(x)$? *Solution:* The smallest value attained by $q(x)$ is 8 since $4(x - 2)^2 \geq 0$ for all x.

4. The temperature $f(t)$ at a desert location varies sinusoidally from a low of 40°F at $t = 0$ hours (midnight) to a high of 80°F at $t = 12$ hours (noon). (See graph in Figure 3, which shows the temperature over two complete days.)

   (a) (6) What is the period of this sinusoidal oscillation? *Answer:* 24 hours

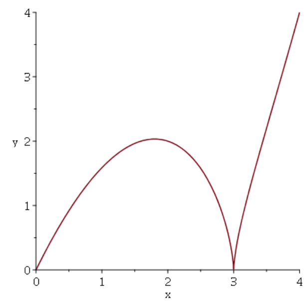

Figure 1: The graph $y = f(x)$

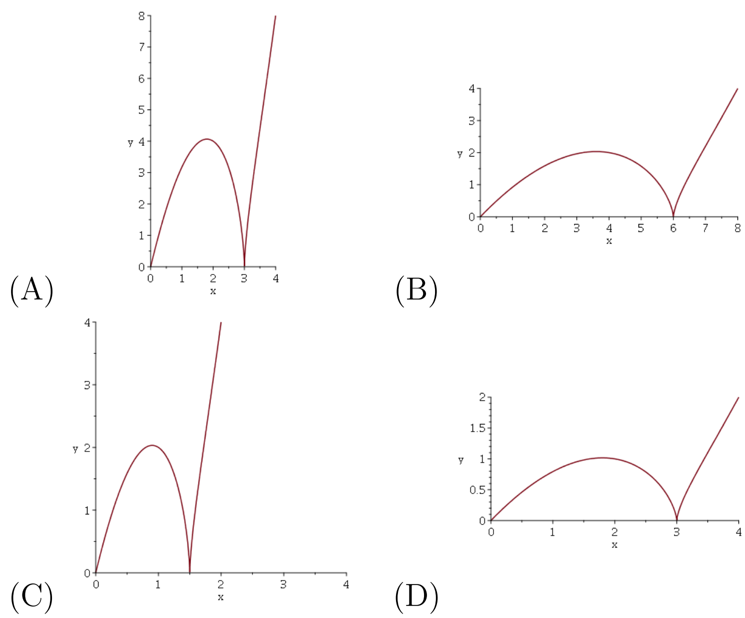

Figure 2: The transformed graphs

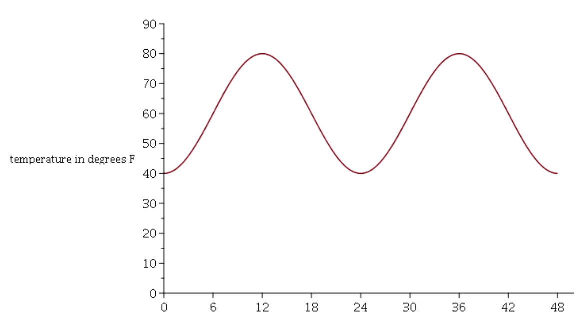

Figure 3: Graph of temperature as a function of time.

(b) (6) What is the amplitude? *Answer:* $\frac{1}{2}(80 - 40) = 20$

(c) (8) Give a possible formula for f as a function of t.

Solution: Since the period starts at one of the minimum values, this looks like a cosine graph, with horizontal and vertical scaling, plus a vertical shift and reflection across the x-axis. The vertical shift can be found from the difference between the maximum and the amplitude: $80 - 20 = 60$. One form is

$$f(t) = -20\cos\left(\frac{2\pi t}{24}\right) + 60.$$

Another possibility (with horizontal shifting too by 6 hours to the right) would be

$$f(t) = 20\sin\left(\frac{2\pi(t-6)}{24}\right) + 60.$$

0

5. You are traveling by horse along a straight line road starting from $x = 0$ (miles) at time $t = 0$ (hours). For the first hour, you move in the positive x-direction at 5 miles per hour. At $t = 2$, you realize you have dropped an important item from your saddle bag. So you turn around and retrace your steps to 5 miles per hour. You find the item at $t = 3$. Then you turn back around and continue at 5 miles per hour for an additional 2 hours.

   (a) (10) Sketch the graph of your position x as a function of time t for $0 \le t \le 5$.

   Solution: The graph is shown in Figure 4 on the next page.

   (b) (10) Give your position x as a piecewise-defined function of t.

Solution: Using the point-slope form for equation of lines, this is the function

$$x(t) = \begin{cases} 5t & \text{if } 0 \le t \le 2 \\ -5t + 20 & \text{if } 2 \le t \le 3 \\ 5t - 10 & \text{if } 3 \le t \le 5. \end{cases}$$

6. Extra Credit. (10) A sample of a radioactive element is decaying over time. The mass present at time t is given by according to $M(t) = 139e^{-0.003t}$ grams, where t is in months. When will the mass present reach 50 grams?

Solution: We want to solve the equation

$$50 = 139e^{-0.003t}$$

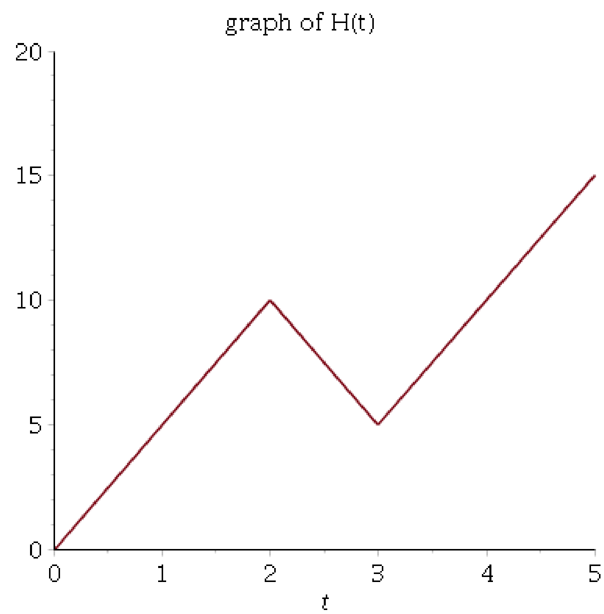

Figure 4: Figure for Question 6 (a)

for t. Divide by 139 first to get

$$e^{-0.003t} = \frac{50}{139}.$$

Then taking natural logs of both sides gives

$$-0.003t = \ln\left(\frac{50}{139}\right) \Rightarrow t = \frac{\ln\left(\frac{50}{139}\right)}{-0.003} \doteq 340.8 \text{ months.}$$