MATH 133 – Calculus with Fundamentals 1
Problem Set 8, Part B Solutions – December 1, 2017

Section 4.4

22. Assuming the graph is $y = f(x)$, there is just one inflection point at $x = c$. The graph of $f(x)$ is concave down on $(0, c)$ and concave up on (c, g).

23. Now assuming the graph is $y = f'(x)$, there are inflection points for $y = f(x)$ at $x = b$ and $x = e$. The graph of $f(x)$ is concave up on $(0, b)$ and (e, g) (the intervals where $f'(x)$ is increasing) and the graph of $f(x)$ is concave down on (b, e) (the interval where $f'(x)$ is decreasing).

24. Now assuming the graph is $y = f''(x)$, there inflection points for $y = f(x)$ at the points where $f''(x)$ changes sign, so at $x = a, d, f$. The graph $y = f(x)$ is concave up on the intervals where $f''(x) > 0$ (that is, where the graph $y = f''(x)$ is above the x-axis), so (a, d) and (f, g). The graph $y = f(x)$ is concave down on the intervals where $f''(x) < 0$ (that is, where the graph $y = f''(x)$ is below the x-axis), so $(0, a)$ and (d, f).

Section 4.7

37. The blue rectangle in the figure is fixed with sides of length H, L. The red rectangle depends on the angle θ shown, and has area determined like this. First look at the triangle at the top. This is a right triangle with hypotenuse equal to L and the angle θ as marked. Hence the longer leg as shown (the adjacent side) has length $L\cos(\theta)$, while the other one has length $L\sin(\theta)$. The triangle at the bottom is congruent to this one, so it has the same side lengths. In the same way, the smaller triangles on the sides have hypotenuse H and legs $H\cos(\theta)$ and $H\sin(\theta)$. Now the whole red rectangle is dissected into four triangles and the blue rectangle, so its total area is

$$A = HL + 2 \cdot \frac{1}{2}L^2\cos(\theta)\sin(\theta) + 2 \cdot \frac{1}{2}H^2\cos(\theta)\sin(\theta).$$

This simplifies to

$$A = HL + (L^2 + H^2)\cos(\theta)\sin(\theta).$$

To find the maximum area, we differentiate with respect to θ, set $= 0$ and solve:

$$\frac{dA}{d\theta} = (L^2 + H^2)(\cos^2(\theta) - \sin^2(\theta)) = 0$$

when $\cos^2(\theta) = \sin^2(\theta)$, or $\cos(\theta) = \pm\sin(\theta)$. It is clear from the diagram that only $\theta \in [0, \pi/2]$ are reasonable solutions, so there is just one relevant critical point, namely the solution of $\cos(\theta) = \sin(\theta)$ in that interval: $\theta = \frac{\pi}{4}$. This gives a maximum of the area A by the First Derivative Test: If $0 < \theta < \pi/4$, then $\cos(\theta) > \sin(\theta)$, so $\frac{dA}{d\theta} > 0$, while if $\pi/4 < \theta < \pi/2$, $\cos(\theta) < \sin(\theta)$ and $\frac{dA}{d\theta} < 0$. (You could also use the Second Derivative Test or the A function values at $\theta = 0, \pi/4, \pi/2$ to get the same conclusion.) Hence the maximum area is

$$A = HL + (L^2 + H^2)\cos(\pi/4)\sin(\pi/4) = HL + \frac{1}{2}(L^2 + H^2).$$