College of the Holy Cross, Fall Semester, 2017
MATH 133, Solutions for Midterm 4
Thursday, December 7

1. Find $\frac{dy}{dx}$; do not simplify:

   A) (7.5) $y = \frac{\ln(x)+x^2}{\cos(x)}$
   Solution: By the Quotient Rule and the derivative rules for cos and ln,
   $$\frac{dy}{dx} = \frac{\cos(x)(1/x + 2x) - (\ln(x) + x^2)(-\sin(x))}{\cos^2(x)}.$$

   B) (7.5) $y = \sin^{-1}(\sqrt{x})$
   Solution: By the Chain Rule,
   $$\frac{dy}{dx} = \frac{1}{\sqrt{1-(\sqrt{x})^2}} \cdot \frac{1}{2\sqrt{x}}.$$

   C) (10) $x^3y^2 - 4y + x^4 = 0$ (use implicit differentiation)
   Solution: We have
   $$x^3 \cdot 2y\frac{dy}{dx} + 3x^2y^2 - 4\frac{dy}{dx} + 4x^3 = 0$$
   So
   $$\frac{dy}{dx} = \frac{-3x^2y^2 - 4x^3}{2x^3y - 4}.$$

2. (20) A stationary observer watches a weather balloon being launched from a point 600 feet away from her position. The line of sight distance from the observer to the balloon is changing at a rate of 20 feet per second. How fast is the height of the balloon changing when the balloon is 400 feet above the ground?

   Solution: Let z be the distance from the observer to the balloon, and y be the height of the balloon above the ground (both functions of time). We have $z^2 = 600^2 + y^2$ by the Pythagorean Theorem. Hence differentiating with respect to time:
   $$2z\frac{dz}{dt} = 2y\frac{dy}{dt}.$$
   At the given time $y = 400$, so $z = \sqrt{600^2 + 400^2} \doteq 721.1$. We are given $\frac{dz}{dt} = 20$, so
   $$2 \cdot 721.1 \cdot 20 = 2 \cdot 400 \cdot \frac{dy}{dt}.$$
   Then
   $$\frac{dy}{dt} = \frac{721.1 \cdot 20}{400} \doteq 36.1$$
   (feet per second.)

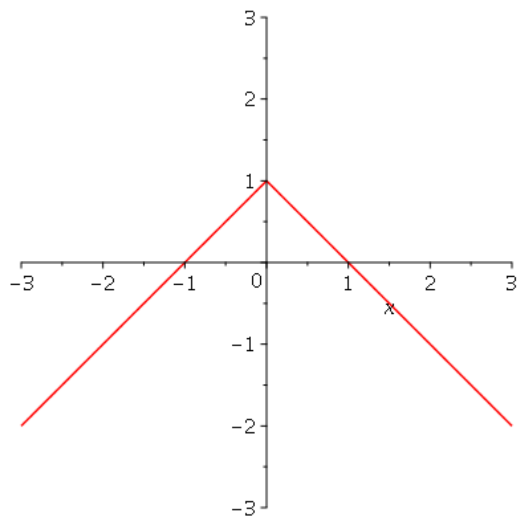

Figure 1: $y = f'(x)$ for Problem 3

3. All parts of this question refer to the plot in Figure 1, *which is* $y = f'(x)$ *for some function* $f(x)$. Assume the whole domain of the functions $f(x)$ and $f'(x)$ is the interval $[-3, 3]$ shown.

   (A) (10) Find the critical points of $f(x)$ in the interval shown:

   Answer: $x = -1, 1$ (where the graph of $f'(x)$ crosses the x-axis.

   (B) (5) Briefly, in your own words, state how the First Derivative Test distinguishes between local maxima, local minima, and critical points that are neither:

   Answer: If the sign of f' changes from $+$ to $-$ at the critical point, then f has a local maximum there; if the sign of f' changes from $-$ to $+$, then f has a local minimum; if the sign of f' does not change, then f has neither a local maximum nor a local minimum.

   (C) (5) Identify each of the points you found in part (A) as a local maximum, local minimum, or neither:

   Answer: f has a local minimum at $x = -1$; f has a local maximum at $x = 1$.

   (D) (5) Find all the inflection points of $f(x)$.

   Answer: Concavity of f changes at $x = 0$, where f' changes from increasing to decreasing.

   (E) (5) Over which intervals is $y = f(x)$ concave up? concave down?

   Concave up: $x < 0$ where f' is increasing; Concave down: $x > 0$ where f' is decreasing.

4. A rectangular poster is to have total area 600 square inches, including blank 1 inch wide margins on all four sides of a central printed area. What overall dimensions will maximize the printed area? The parts of this problem will lead you to the answer.

(A) (4) Draw a diagram clearly showing the whole poster, the central printed area, and the 1 inch wide blank margins.

Solution: Appropriate diagram showing one rectangle inside another one with 1-inch margins on all four sides of the inner regions is OK.

(B) (4) Call the horizontal side of the whole poster x and the vertical side of the whole poster y.

Solution: Area of whole poster is $xy = 600$; area of printed region is $AP = (x-2)(y-2)$ (subtract one inch for the margins on top and bottom, one inch for the margins on left and right).

(C) (4) Solve for y in terms of x using the area of the whole poster and substitute into the area of the printed region.

Solution: $y = \dfrac{600}{x}$. So

$$AP = (x-2)\left(\frac{600}{x} - 2\right) = 604 - \frac{1200}{x} - 2x.$$

(D) (5) Determine a critical point of the area function.

Solution: $AP'(x) = \dfrac{1200}{x^2} - 2 = 0$ when $x^2 = 600$, or $x = 10\sqrt{6} \doteq 24.49$.

(E) (4) How do you know your critical point is a maximum of the area?

Solution: $AP'' = \dfrac{-2400}{x^3}$. When you plug in $x = 10\sqrt{6}$, the result is negative, so $AP(x)$ has a maximum there by the Second Derivative Test. This is the only critical point, so it must be an overall maximum too.

(F) (4) What are the dimensions x and y of the poster of maximum printed area?

Solution:

$$x = \sqrt{600} \doteq 24.49; \qquad y = \frac{600}{\sqrt{600}} = \sqrt{600}.$$

The poster of maximum area is a square.