

MATH 133 – Calculus 1 with Fundamentals
Second Derivative and Concavity
November 27, 2017

Background

We say f (or the graph $y = f(x)$) is *concave up* on an interval if f' is increasing on that interval, and similarly, f or its graph is *concave down* if f' is decreasing on that interval. Combined with our results from before Thanksgiving, this says:

- If $f''(x) > 0$ on an interval, then f or its graph is concave up on that interval
- If $f''(x) < 0$ on an interval, then f or its graph is concave down on that interval
- A point $(c, f(c))$ on the graph of f where the concavity changes is called a *point of inflection* of f .

The notion of concavity can also be used to state a second method for determining whether critical points are local maxima or local minima, called the Second Derivative Test:

Theorem 1 (*Second Derivative Test*) *Let f be differentiable on some open interval containing a critical point c . In addition, assume $f''(c)$ exists.*

- (a) *If $f''(c) > 0$, then $f(c)$ is a local minimum*
- (b) *If $f''(c) < 0$, then $f(c)$ is a local maximum*
- (c) *If $f''(c) = 0$, there is no conclusion.*

In the last case here, f could have either a local maximum or a local minimum, or neither, so no conclusion is possible. *Technical Comment:* In the other cases, the intuition is that f' should be increasing or decreasing on an interval containing c depending on the sign of $f''(c) = (f')'(c)$, so that (a) corresponds to a case where the graph is concave up at c and (b) corresponds to a case where the graph is concave down at c . This would follow, for instance, if we knew (in addition) that f'' was continuous on some interval containing c .

Questions

1. Consider the graph $f(x) = x^3 - 3x^2 + 2x$ on the interval $[-1, 3]$ from last time (the plot is also on the back of this sheet). Find the intervals where f is concave up and the intervals where f is concave down. How many points of inflection are there on this graph and where are they located?
2. Consider $f(x) = x^2e^{-x}$.
 - (a) Compute $f'(x)$ and find all critical points.
 - (b) Determine the sign of $f'(x)$ on each interval between successive critical points, and use that to classify the critical points as local maxima or local minima by the First Derivative Test.
 - (c) Now compute $f''(x)$ and check your answers in (b) by using the Second Derivative Test.

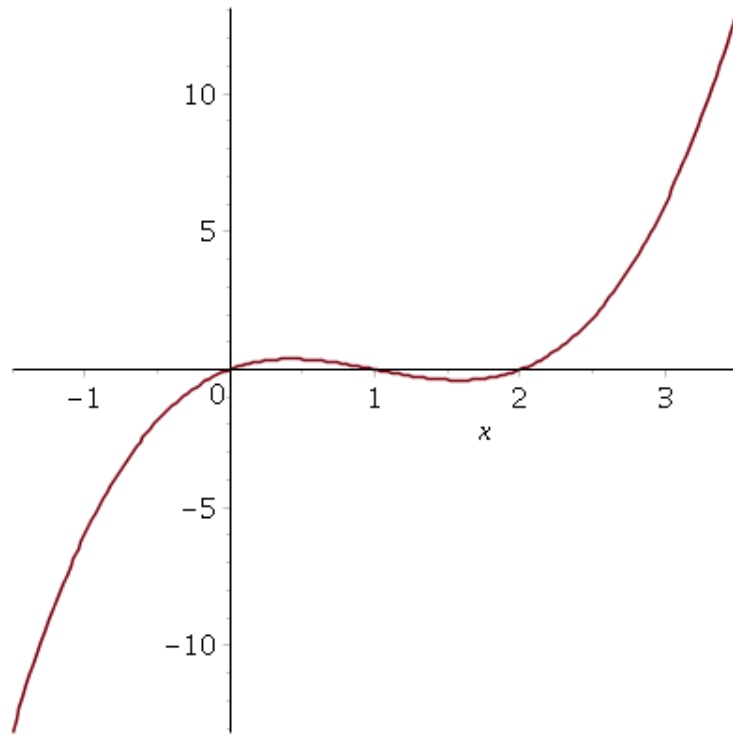


Figure 1: Plot for question 1

- (d) Determine all points of inflection of f .
3. Repeat question 2 for $f(x) = 2x^4 - 3x^2 + 2$.