

MATH 133 – Calculus with Fundamentals 1  
The Chain Rule and Implicit Differentiation  
November 6, 2017

*Background*

As we saw in today's video, we may want to compute tangent lines to curves that are more general than graphs  $y = f(x)$ . (Equivalently we may want to understand rates of change of two related quantities when the relationship is more complicated than saying one is a function of the other.) For instance, consider the curve defined by the equation

$$x^3 + y^3 - 3xy = 0, \tag{1}$$

plotted on the back of this sheet. Note that this curve is *not a single graph*  $y = f(x)$  because *it fails the vertical line test(!)*. However, by focusing on just portions of the curve, we can see that there are several graphs that lie on the curve. Each of those defines  $y$  *implicitly as a function of*  $x$  on a portion of the curve. These are called *implicit functions* because we will not have explicit formulas for them. However we can do things like compute derivatives because in (1), we can *think of  $y$  as a function of  $x$*  and differentiate things like  $y^3$  by the chain rule, and things like the  $xy$  using the product rule. If we do this we get a new equation:

$$3x^2 + 3y^2 \frac{dy}{dx} - 3x \frac{dy}{dx} - 3y = 0.$$

Solving for  $\frac{dy}{dx}$ , we get

$$\frac{dy}{dx} = \frac{y - x^2}{y^2 - x}.$$

This process is called *implicit differentiation*. Then, for instance, if we wanted to find points where the tangent line was horizontal, we would set  $\frac{dy}{dx} = 0$ , and get  $y = x^2$ . So from (1),

$$x^3 + x^6 - 3x^3 = 0 \quad \text{or} \quad x^3(x^3 - 2) = 0.$$

This is satisfied when  $x = 0$  and also  $x = 2^{1/3} \doteq 1.26$ . The point  $(2^{1/3}, 2^{2/3})$  is one such point!

*Questions*

1. Find  $\frac{dy}{dx}$  by implicit differentiation, then performed the other indicated calculations (if any):
  - (a)  $3x^2 + 2y^2 = 5$ . Use  $\frac{dy}{dx}$  to find the equation of the tangent line to this curve (an ellipse) at  $(1, 1)$ .
  - (b)  $\sin(xy) = x$ .
  - (c)  $xy + x^2y^2 = 6$ . Find the equation of the tangent line to this curve at the point  $(2, 1)$ .
2. Find all the points on the curve defined by  $y^2 = x^3 - 3x + 1$  where the tangent line is *horizontal*. (This curve is shown in Figure 10 on page 173 of our book.)

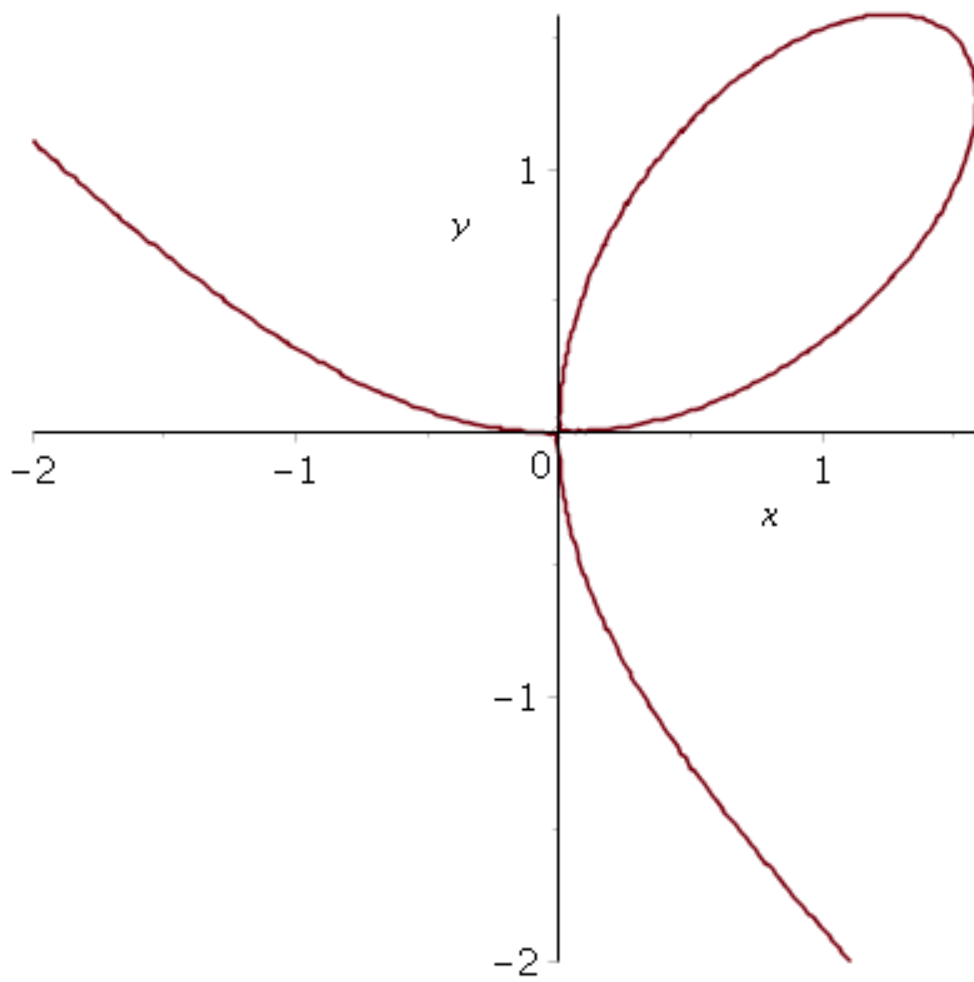


Figure 1: The curve  $x^3 + y^3 - 3xy = 0$  (the “folium of Descartes”)