MATH 133 – Calculus with Fundamentals 1
The Derivative Chain Rule
November 2, 2017

Background

Our next major derivative short-cut rule is *one of the most important.* This rule, called the *Chain Rule* allows us to differentiate functions that are built up by *composition.* Here's what it says: If g is differentiable at x and f is differentiable at $g(x)$, then the composition $(f \circ g)(x) = f(g(x))$ is differentiable at x and

$$\frac{d}{dx} f(g(x)) = f'(g(x))g'(x).$$

In words: The derivative of the composition is *the derivative of the outside function (i.e. f'), with $g(x)$ "plugged in," times the derivative of g.* For example, the function

$$h(x) = \sqrt{x^2 + 4x + 9}$$

is the composition $f(g(x))$ where $f(x) = \sqrt{x}$ and $g(x) = x^2 + 4x + 9$ is "plugged in." The Chain Rule says the derivative of h will be given by computing $f'(x) = \frac{1}{2\sqrt{x}}$ (do you see where that comes from?), plugging g into $f'(x)$, then multiplying by $g'(x)$:

$$h'(x) = \frac{1}{2\sqrt{x^2 + 4x + 9}} \cdot (2x + 4) = \frac{x + 2}{\sqrt{x^2 + 4x + 9}}.$$

Today's class will be devoted to understanding and practicing this rule. We'll continue and use this a different way on Monday.

Questions

For each function, identify an $f(x)$ and $g(x)$ such that the given function is the composition $f(g(x))$. Then apply the Chain Rule and compute the derivative:

(a)
$$h(x) = e^{3x+1}.$$

(b)
$$h(x) = \frac{1}{(x^4 + 5x^2 + 1)^{3/2}}.$$

(c)
$$h(x) = \sin(\cos(x) + x).$$

(d)
$$h(x) = (\tan(x) + 4x)^3.$$

(e) Sometimes we need to use the Chain Rule more than once (if the function we're looking at is "several composition layers deep" like

$$h(x) = \cos^2(4x^3 + 2) = (\cos(4x^3 + 2))^2.$$

Note that this is $f(g(x))$ with $f(x) = x^2$ and $g(x) = \cos(4x^3 + 2)$. But $g(x)$ is also a composition, so you'll need to use the Chain Rule again to find $g'(x)$. With these hints, find $h'(x)$.