

MATH 133 – Calculus with Fundamentals 1
Indeterminate Forms and Algebraic Evaluation of Limits
October 3, 2017

Background

Most of the limits we will need to deal with in calculus are so-called *indeterminate forms* like $0/0$, ∞/∞ (and eventually also $0 \cdot \infty$ and $\infty - \infty$), where the Limit Laws and continuity don't help immediately. For instance

$$\lim_{x \rightarrow 1} \frac{x^4 - x^3}{x^2 - 1}$$

is a $0/0$ form since the numerator and denominator of this rational function both tend to 0 as $x \rightarrow 1$. The perhaps surprising thing is that in cases like this we can often

- (1) algebraically cancel a factor producing the 0's in the numerator and denominator, and/or
- (2) rearrange the function in other ways

to see that a limit exists and evaluate it. For example, the limit above can be evaluated like this. After factoring and cancelling a common factor of $x - 1$, we get:

$$\frac{x^4 - x^3}{x^2 - 1} = \frac{x^3(x - 1)}{(x + 1)(x - 1)} = \frac{x^3}{x + 1}.$$

This new function is equal to the original one for all $x \neq 1$, but it is continuous at $x = 1$. Hence we can see

$$\lim_{x \rightarrow 1} \frac{x^4 - x^3}{x^2 - 1} = \lim_{x \rightarrow 1} \frac{x^3}{x + 1} = \frac{1}{2}.$$

This method will always involve replacing the original function by some other function that is defined and continuous at the x -value where we are taking the limit, and then the limit can be obtained by substitution, *after the replacement*.

The following questions will “lead you through” some first examples. *Eventually, though, you will need to be able to recognize which algebraic methods apply and work through examples like this without the prompts from me.*

Questions

- (1) Consider the limit

$$\lim_{x \rightarrow 4} \frac{x^3 - 64}{x^2 - 16},$$

- (a) First explain why this is a $0/0$ form.
- (b) Next, factor the top and bottom. You should find a common factor of $x - 4$ that can be cancelled.

- (c) Do this, and then note that “what is left” is continuous at $x = 4$, so we can find the limit by substitution in the cancelled form. What is the limit?

- (2) Next, consider this limit:

$$\lim_{x \rightarrow 3} \frac{\sqrt{x^2 + 16} - 5}{x - 3}.$$

- (a) Explain why this is also a $0/0$ form.
- (b) We can look at the top as $\sqrt{x^2 + 16} - \sqrt{25}$. In cases like this, multiply top and bottom by the conjugate radical $\sqrt{x^2 + 16} + \sqrt{25}$ and simplify the top.
- (c) You should now see something to cancel so that the remaining function is continuous at $x = 3$. Do that and determine the limit.
- (3) Next consider this limit:

$$\lim_{x \rightarrow 5} \frac{\frac{1}{x^2} - \frac{1}{25}}{x - 5}$$

- (a) Why is this one an indeterminate $0/0$ form?
- (b) Put the two terms on the top over a common denominator. Then simplify the fraction and find something to cancel.
- (c) Evaluate the limit.
- (4) Many ∞/∞ indeterminate form limits can be evaluated in a similar way. We’ll consider

$$\lim_{x \rightarrow \pi/2} \frac{\tan(x)}{\sec(x)}$$

- (a) Explain why this one is an ∞/∞ indeterminate form.
- (b) Rewrite in terms of $\sin(x)$ and $\cos(x)$ and simplify, cancelling something in the process.
- (c) What is left should be continuous at $x = \pi/2$, so evaluate the limit by substitution.