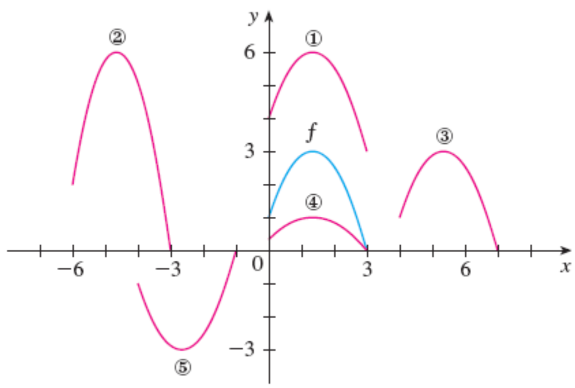
College of the Holy Cross
MATH 133, Calculus With Fundamentals 1
Solutions for Final Examination – Friday, December 15

I. The graph $y = f(x)$ is given in light blue. Match each equation with one of the numbered magenta graphs.



(5) A) $y = 2f(x+6)$ is plot number: ________

Solution: 2 (shifted left and stretched vertically)

(5) B) $y = \frac{1}{3}f(x)$ is plot number: ________

Solution: 4 (compressed vertically)

(5) C) $y = f(x) + 3$ is plot number: ________

Solution: 1 (shifted vertically)

(5) D) $y = f(x-4)$ is plot number: ________

Solution: 3 (shifted right)

(5) E) $y = -f(x+4)$ is plot number: ________

Solution: 5 (reflected across x-axis and shifted left)

II. A cup of hot chocolate is set out on a counter at $t = 0$. The temperature of the chocolate t minutes later is $C(t) = 65 + 55e^{-t/4}$ (in degrees F).

A) (5) What is the temperature of the chocolate at $t = 0$?

Solution: $C(0) = 65 + 55 \cdot 1 = 120$ degrees F.

B) (10) What is the rate of change of the temperature at $t = 10$ minutes?

Solution: The rate of change is

$$C'(10) = 55e^{-10/4} \cdot (-1/4) \doteq -1.13$$

(degrees per minute)

C) (10) How long does it take for the temperature to reach $100°F$?

Solution: Solve the equation $100 = 65 + 55e^{-t/4}$ for t:

$$t = -4\ln(35/55) \doteq 1.81$$

(minutes).

III. Compute the following limits. Any legal method is OK.

(A) (10) $\lim_{x\to 3} \frac{x^2 + x - 12}{x^2 - 5x + 7}$.

Solution: This one is not an indeterminate form, since the denominator is not zero at $x = 3$. By the Limit Laws, the limit is

$$= \frac{3^2 + 3 - 12}{3^2 - 15 + 7} = 0.$$

(B) (10) $\lim_{x\to 1} \frac{x-1}{x^2-1}$.

Solution: This one is an indeterminate form, but we can factor the bottom and cancel the $x - 1$ to get

$$\lim_{x\to 1} \frac{x-1}{x^2-1} = \lim_{x\to 1} \frac{1}{x+1} = \frac{1}{2}.$$

(C) (10) $\lim_{x\to 0} \frac{\sqrt{x+1}-1}{x}$

Solution: Multiply the top and the bottom by the conjugate radical, simplify the top, cancel the x between the top and the bottom, then evaluate:

$$\begin{aligned}
\lim_{x\to 0} \frac{\sqrt{x+1}-1}{x} &= \lim_{x\to 0} \frac{\sqrt{x+1}-1}{x} \cdot \frac{\sqrt{x+1}+1}{\sqrt{x+1}+1} \\
&= \lim_{x\to 0} \frac{(x+1)-1}{x(\sqrt{x+1}+1)} \\
&= \lim_{x\to 0} \frac{1}{\sqrt{x+1}+1} = \frac{1}{2}.
\end{aligned}$$

(D) (10) $\lim_{x\to 0} \frac{\tan(x)}{x}$

Solution: Since $\tan(x) = \sin(x)/\cos(x)$, the limit product law says:

$$= \lim_{x\to 0} \frac{\sin(x)}{x} \cdot \lim_{x\to 0} \frac{1}{\cos(x)} = 1 \cdot 1 = 1.$$

IV. A) (10) *Using the limit definition*, and showing all necessary steps to justify your answer, compute $f'(x)$ for $f(x) = 5x^2 + x$.

Solution:

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h\to 0} \frac{5(x+h)^2 + (x+h) - 5x^2 - x}{h} \\ &= \lim_{h\to 0} \frac{5x^2 + 10xh + 5h^2 + x + h - 5x^2 - x}{h} \\ &= \lim_{h\to 0} \frac{10xh + h + 5h^2}{h} \\ &= \lim_{h\to 0} 10x + 1 + 5h \\ &= 10x + 1. \end{aligned}$$

IV. (continued) Using appropriate derivative rules, compute the derivatives of the following functions. You do not need to simplify your answers.

B) (5) $g(x) = 4x^3 + \sqrt{x} + \frac{2}{\sqrt[4]{x}} + e^2$

Solution: Write $\sqrt{x} = x^{1/2}$ and $1/\sqrt[4]{x} = x^{-1/4}$. By the power rule:

$$g'(x) = 12x^2 + \frac{1}{2}x^{-1/2} - \frac{1}{2}x^{-5/4}$$

C) (10) $h(x) = \frac{\cos(x) + x}{\sin(x)}$

Solution: By the Quotient Rule,

$$h'(x) = \frac{\sin(x)(-\sin(x) + 1) - (\cos(x) + x)\cos(x)}{\sin^2(x)}.$$

D) (10) $i(x) = \ln(x^3 + e^x)$

Solution: By the derivative rule for $\ln(u)$ (with the chain rule),

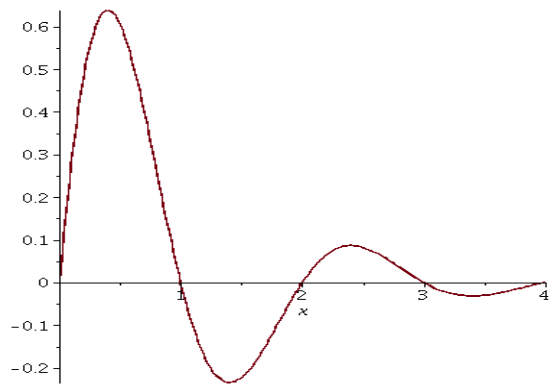
$$i'(x) = \frac{1}{x^3 + e^x} \cdot (3x^2 + e^x).$$

E) (10) $j(x) = \tan^{-1}(3x + 7)$

Solution: By the derivative rule for the inverse tangent and the chain rule,

$$j'(x) = \frac{1}{1 + (3x + 7)^2} \cdot 3.$$

V. The following graph shows the *derivative* $f'(x)$ for some function $f(x)$ defined on $0 \leq x \leq 4$. Note: This *is not* $y = f(x)$, it is $y = f'(x)$.



Using the graph, *estimate*

A) (5) The interval(s) on which $f(x)$ is increasing.

Solution: These are the intervals where $f'(x) > 0$ so $(0, 1)$ and $(2, 3)$.

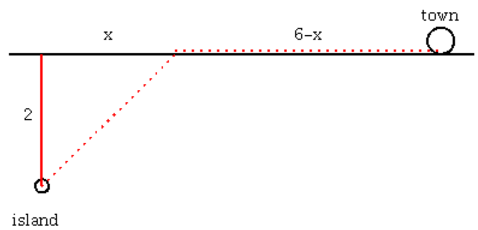
B) (10) The critical numbers of $f(x)$ *in the open interval* $(0, 4)$. Say what the behavior of $f(x)$ is at each critical number (local max, local min, neither)

Solution: In the open interval $(0, 4)$ f has critical points at $x = 1, 2, 3$. f' changes sign from $+$ to $-$ at $x = 1, 3$, so those are local maxima; f' changes sign from $-$ to $+$ at $x = 2$, so that is a local minimum. (All of this is based on the first derivative test.)

C) (10) The interval(s) on which $y = f(x)$ is concave down.

Solution: $f(x)$ is concave down on intervals where $f'(x)$ is decreasing, so roughly $(0.4, 1.4)$ and $(2.4, 3.4)$.

VI. A town wants to build a pipeline from a water station on a small island 2 miles from the shore of its water reservoir to the town. One possible route is shown dotted in red. The town is 6 miles along the shore from the point nearest the island. It costs \$3 million per mile to lay pipe under the water and \$2 million per mile to lay pipe along the shoreline.



A) (5) Give the cost $C(x)$ of constructing the pipeline as a function of x.

Solution: By the Pythagorean theorem, the distance over water is $\sqrt{4+x^2}$ and the distance over land is $6-x$. By the given information about cost per mile, we have

$$C(x) = 3\sqrt{4+x^2} + 2(6-x)$$

(where the cost is given in units of millions of dollars).

B) Differentiate your cost function to find $C'(x)$.

Solution: The derivative is

$$3 \cdot \frac{1}{2} \cdot (4+x^2)^{-1/2} \cdot 2x - 2 = \frac{3x}{\sqrt{4+x^2}} - 2.$$

C) Find all the critical points of $C(x)$.

Solution: The function $C(x)$ has a critical number for $x > 0$ at the positive solution of $C'(x) = 0$:

$$\begin{aligned} 0 &= \frac{3x}{\sqrt{4+x^2}} - 2, \text{ or} \\ 3x &= 2\sqrt{4+x^2} \\ 9x^2 &= 16 + 4x^2 \\ 5x^2 &= 16 \\ x &= \frac{4}{\sqrt{5}} \doteq 1.79. \end{aligned}$$

There is another critical point at $x \doteq -1.79$, but that would clearly give a longer pipeline costing more, so we can ignore it.

D) Where on the shoreline should the pipeline hit land to minimize the costs of construction? Say how you know this gives the minimum cost.

Solution: To find the minimum of $C(x)$, we can restrict to x in the closed interval $[0, 6]$, since it clearly does no good to take $x < 0$ or $x > 6$. We have $C(0) = 18$, $C(6) = 3\sqrt{40} \doteq 19.0$, and $C\left(\frac{4}{\sqrt{5}}\right) \doteq 16.47$. So the minimum cost is attained at $x = \frac{4}{\sqrt{5}} \doteq 1.79$ miles. Alternatively, one could use the First or Second Derivative Test to show this is a local minimum and then argue from the fact that there is only one critical point for $x > 0$.

VII. (15) A block of dry ice (solid CO_2) is evaporating and losing volume at the rate of 10 cm^3/min. It has the shape of a cube at all times. How fast are the edges of cube shrinking when the block has volume 216 cm^3?

Solution: Call the side of the cube x. Then $V = x^3$. Taking time derivatives, we have $\frac{dV}{dt} = 3x^2 \frac{dx}{dt}$. From the given information, when $V = 216$, $x = 6$ and $\frac{dV}{dt} = -10$. Therefore the rate of change of the side of the cube is

$$\frac{dx}{dt} = \frac{-10}{3 \cdot 6^2} = \frac{-5}{54} \doteq -.093$$

(units cm/min). The side of the cube is decreasing at about .09 cm/min.