MATH 133 – Calculus with Fundamentals 1
Discussion Day – Related Rates Problems – Solutions
November 16 and 17, 2015

Questions

(1) A bicyclist moves counterclockwise around an elliptical track in the shape of the curve with equation:

$$\frac{x^2}{1600} + \frac{y^2}{900} = 1, \tag{1}$$

where x and y are distances in meters.

(a) In which of the four quadrants of this xy-coordinate system is $\frac{dx}{dt} > 0$? In which quadrants is $\frac{dy}{dt} < 0$? Explain. Recall that the cyclist is moving counterclockwise around the ellipse.

Solution: Since the cyclist is moving *counterclockwise*, the x-coordinate of the position is increasing (that's what $\frac{dx}{dt} > 0$ indicates) in quadrants III and IV. Similarly, the y-coordinate of the position is decreasing ($\frac{dy}{dt} < 0$) in quadrants II and III.

(b) Find a relation between $\frac{dx}{dt}$ and $\frac{dy}{dt}$ that holds at all times. (Hint: Differentiate (1) with respect to t.)

Solution: Taking $\frac{d}{dt}$ in equation (1) above, we get (chain rule):

$$\frac{1}{1600} 2x \frac{dx}{dt} + \frac{1}{900} 2y \frac{dy}{dt} = 0.$$

(c) When the bicyclist moves through the point $(x, y) = (24, 24)$, the y-coordinate of her position is increasing at 3 meters per second. How fast is the x-coordinate of her position changing?

Solution: In the relation from part (b), substitute $x = y = 24$ and $\frac{dy}{dt} = 3$:

$$\frac{1}{1600} 2 \cdot 24 \frac{dx}{dt} + \frac{1}{900} 2 \cdot 24 \cdot 3 = 0.$$

So solving for $\frac{dx}{dt}$, we get

$$\frac{dx}{dt} = \frac{-1600 \cdot 3}{900} \doteq -5.33\text{m/sec}.$$

(The minus sign indicates that x is decreasing.)

(2) Sand is being poured onto a sand pile at a constant rate of 4 cubic centimeters per second. The pile has the shape of a right circular cone with $h = \frac{r}{2}$ at all times, where h is the height, and r is the base radius. How fast is the height changing when the height is 10 centimeters? And how fast is the circumference of the base changing at that time? (Note: The volume of a cone is $V = \frac{\pi r^2 h}{3}$.)

Solution: For the first part, since $r = 2h$ at all times, the volume of the pile of sand can be written as

$$V = \frac{4\pi h^3}{3},$$

where h and V are functions of $t =$ time. Now we differentiate with respect to t, using the chain rule on the h^3:

$$\frac{dV}{dt} = 4\pi h^2 \frac{dh}{dt}$$

Then sinse we are given $\frac{dV}{dt} = 4$ cubic centimeters per second and $h = 10$,

$$4 = 4\pi 10^2 \frac{dh}{dt} \Rightarrow \frac{dh}{dt} = \frac{1}{100\pi} \doteq .0032\text{m/sec}.$$

For the second part, the circumference of the base of the cone is $C = 2\pi r = 4\pi h$. So

$$\frac{dC}{dt} = 4\pi \frac{dh}{dt} = 4\pi \cdot \frac{1}{100\pi} = \frac{1}{25} = .04\text{m/sec}.$$

(3) A radio-controlled drone is tracked by an observer stationed at ground level standing still exactly 40 meters from the point where the drone took off. The drone ascends along a vertical path perpendicular to the ground.

(a) Suppose the height of the drone is increasing at 10 meters per second. How fast is the angle between the horizontal and the direct line of sight to the drone changing when the angle is $\frac{\pi}{4}$?

Solution: The observer, the drone, and the launching point of the drone form a right triangle at all times. Let's call the height of the drone h (this is a function of t). The angle θ between the horizontal and the direct line of sight satisfies

$$\tan(\theta) = \frac{h}{40},$$

so after taking time derivatives and using the chain rule:

$$\sec^2(\theta)\frac{d\theta}{dt} = \frac{1}{40}\frac{dh}{dt}.$$

We are given $\frac{dh}{dt} = 40$ and we want $\frac{d\theta}{dt}$ when $\theta = \frac{\pi}{4}$. So substituting in these values:

$$\begin{aligned} \sec^2\left(\frac{\pi}{4}\right)\frac{d\theta}{dt} &= \frac{1}{40} \cdot 10 \\ \frac{d\theta}{dt} &= \frac{1}{4}\cos^2\left(\frac{\pi}{4}\right) \\ &= \frac{1}{8} = .125\text{rad/sec}. \end{aligned}$$

(b) If the distance from the observer to the drone is changing at 8 meters per second when the height of the drone is 100 meters, how fast is the drone ascending? (Note: the parts here are separate problems; don't use values from one in the other!)

Solution: Let's call the straight line distance from the observer to the drone z and the height h. Then by the Pythagorean theorem:

$$z^2 = 40^2 + h^2.$$

Taking time derivatives on both sides (using the chain rule again)

$$2z\frac{dz}{dt} = 0 + 2h\frac{dh}{dt}.$$

When $h = 100$, the original equation from the Pythagorean theorem says

$$z = \sqrt{40^2 + 100^2} \doteq 108\text{meters}$$

. We are also given $\frac{dz}{dt} = 8$. Then

$$2(108)8 = 2(100)\frac{dh}{dt}$$

so $\frac{dh}{dt} \doteq 8.6$ meters per second.

(4) A laser pointer is placed on a platform that rotates at a constant rate of 20 revolutions per minute. (How many radians per minute is that?) The beam hits a wall 8 meters away from the platform, producing a dot of light that moves horizontally along the wall. Let θ be the angle between the beam and the line through the platform perpendicular to the wall. How fast is the dot of light moving when $\theta = \frac{\pi}{4}$? (Hint: See Figure 10 on page 187 of the text.)

Solution: The geometry of this one is very similar to part (a) of question 3. We have a right triangle formed by the position of the pointer, the closest point on the wall, and the point where the beam hits the wall at all times. Call the distance along the wall from the closest point to the platform x (see the figure from the book).

$$\tan(\theta) = \frac{x}{8}$$

So

$$\sec^2(\theta)\frac{d\theta}{dt} = \frac{1}{8}\frac{dx}{dt}.$$

We are given that the pointer is rotating at 20 revolutions per minute. That means $\frac{d\theta}{dt} = 40\pi$ radians per minute (one full revolution is 2π radians). So when $\theta = \frac{\pi}{4}$ we have $\sec^2(\pi/4) = 2$, so

$$2 \cdot 40\pi \cdot 8 = \frac{dx}{dt}.$$

This gives $\frac{dx}{dt} \doteq 2010.6$ meters per minute.