*Background*

To study a first order differential equation

$$\frac{dy}{dt} = f(t, y)$$

and its solutions qualitatively, we can use the graphical *slope field* associated to the equation. The slope field is a picture obtained by placing at each point $(t, y)$ in the plane a small line segment of slope equal to the number $f(t, y)$. As we will see in a moment, Maple draws these as *arrows*. A solution of the differential equation is then a function $y = g(t)$ whose graph has slope equal to $f(t, g(t))$ at each point $(t, g(t))$ on the graph. That is, the graph *"is tangent to the slope field"* – at each point on the graph of the solution function, the tangent line has the same slope as the slope field arrow at that point.

Today, we will use Maple to draw some slope fields, use them to visualize solutions, and see how numerical methods like Euler's Method apply to give approximate solutions. We will use the logistic equations

$$\frac{dy}{dt} = ky \left(1 - \frac{y}{A}\right)$$

as examples to prepare for work you will be doing in the computer lab next week.

*1. Slope Fields Via Maple*

Executing the commands

```
with(DEtools);
with(plots);
```

once at the start of the session set up the plotting routines used for differential equations and slopefields. Note the capitalization, which is necessary; DE stands for *D*ifferential *E*quation, as you might guess. This command loads an external package of additional Maple commands for working with differential equations. You need to do that to make the `DEplot` command available – the main operation we will be using today. The output will be a list of the commands in the DEtools package.

To plot the slope field for an equation

$$\frac{dy}{dt} = f(t, y)$$

you can enter a command of the following format:

```
DEplot(diff(y(t),t)=f(t,y(t)),y(t),t=a..b,y=c..d);
```

where:
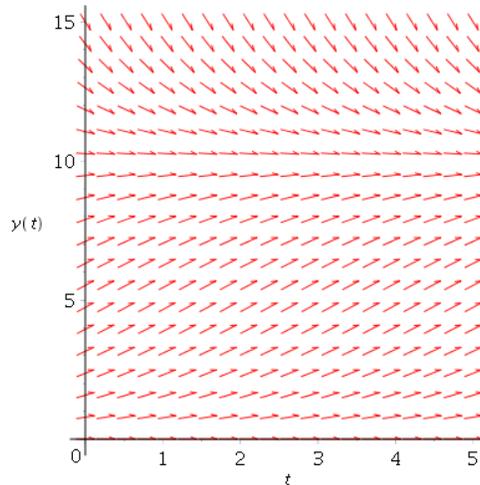
Figure 1: Slope Field for $\frac{dy}{dt} = (.6)y(1 - y/10)$

- The `f` is the expression for the slope function $f(t, y)$. You enter that in usual Maple format (or use a symbolic name for the expression).

- The dependent variable (unknown function) in the differential equation should be entered in the form `y(t)` everywhere. For example, to plot the slope field for the logistic equation $\frac{dy}{dt} = (.6)y(1 - y/10)$, you might enter a command like

    ```
    DEplot(diff(y(t),t) = (0.6)*y(t)*(1 - y(t)/10),y(t),t=0..5,y=0..15);
    ```

- the ranges of $t$- and $y$-values indicate the portion of the plane that will be *plotted* with the slope field. A standard number of grid points is used. The output is shown in Figure 1 above.

*Plotting Solutions in Maple*

It's possible, of course to plot the graph of an analytic solution (i.e. a formula derived by a method like separation of variables) of a differential equation directly using the `plot` command. But even if we have not or can not do that, we can plot *approximate solution graphs* for the solutions of initial value problems for a differential equation, together with the slope field with the same `DEplot` command above, but with different options. For example, to plot the solution of the logistic equation above with $y(0) = 1$, for $0 \leq t \leq 5$, you would enter a command like

```
DEplot(diff(y(t),t) = (0.6)*y(t)*(1-y(t)/10),y(t),t=0..5,y=0..15,[[y(0)=1]],linecolor=black);
```

Note: The default color for the solution curve is a shade of gold that does not show up very well when you print a worksheet on a black and white printer(!). Hence I used the option `linecolor=black`. The general format is:

- The differential equation will always come first,

2
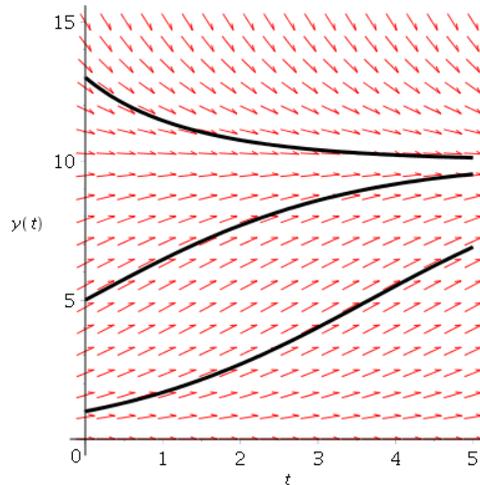
Figure 2: Slope Field for $\frac{dy}{dt} = (0.6)y(1 - y/10)$, together with three solution graphs

- then the dependent variable or unknown function (y(t)),

- then *the range of t-values for which you want to see the graph of the solution*,

- then, in square brackets, separated by commas, a list of initial conditions [y(a)=b].
  In the example above, there's just one, but any number can be included.

If you want to see several solution curves, you can place a *list* of initial conditions, separated by commas, in the [] brackets after the y = 0..15. This plot in Figure 2 shows several such solutions, corresponding to the initial conditions

$$y(0) = 1, y(0) = 5, y(0) = 13$$

*How does Maple do this?*

In case you are wondering, Maple draws these graphs in a particular way, by using an *approximate numerical method similar to numerical integration methods*. There are many such methods; we will briefly discuss one called *Euler's Method* next. The idea is this. Suppose we want approximate values for a solution of $\frac{dy}{dt} = f(t, y)$ and $y(a) = y_0$ for $t$ in some interval $[a, b]$. We can subdivide the interval as before into $n$ equal subintervals with $\Delta t = \frac{b-a}{n}$, proceed as follows:

- "sample" the slope function at $(t_0, y_0)$ to get $f(t_0, y_0)$

- proceed along a straight line with slope $f(t_0, y_0)$ until $t = t_1 = t_0 + \Delta t$. The $y$-coordinate on the line is one approximation for $y(t_1)$. That can be found like this. The line is $y - y_0 = f(t_0, y_0)(t - t_0)$ so when $t = t_1 = t_0 + \Delta t$, we get $y_1 = y_0 + f(t_0, y_0)\Delta t$.

- Then repeat the process starting from $(t_1, y_1)$.

3
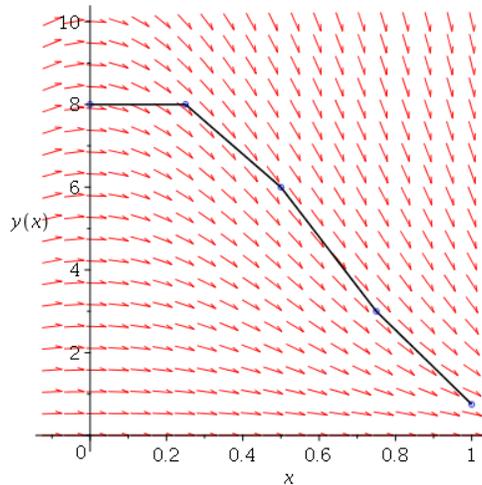
Figure 3: Euler's Method on $\frac{dy}{dt} = -4ty$ with initial condition $y(0) = 8$

- Keep going until you have a table of values for the desired $t$'s, or for some predefined number of steps.

This idea yields the following recipe (called *Euler's Method*) for computing the approximate $y$-values: $y_0$ is given from the initial condition, then

$$t_k = t_{k-1} + \Delta t$$

and

$$y_k = y_{k-1} + f(t_{k-1}, y_{k-1})\Delta t$$

for $k = 1, 2, \ldots, n$.

The following Maple commands will use Euler's Method to generate a table of approximate values of the solution of $y' = -4ty$ with $y(0) = 8$ on $[0, 1]$ with $n = 4$, so $\Delta x = .25$.

```
tl[0]:=0:   yl[0]:=8:
for i to 4 do
    tl[i]:=tl[i-1]+.25;
    yl[i]:= yl[i-1]-4*xl[i-1]*yl[i-1]*(.25);
end do;
```

Then the following commands generate Figure 3 above: they plot the slope field together with the piecewise linear approximate solution graph made by "connecting the dots" for the points computed by Euler's Method:

```
DirField:=DEplot(diff(y(t),t)=-4*t*y(t),[y(t)],t=-.1..1,y=0..10):
Pts := plot([seq([tl[i],yl[i]],i = 0..5)],color=blue,style=point,symbol=circle);
Lines:=plot([seq([tl[i],yl[i]],i=0..4)],color=black):
display(DirField,Pts,Lines);
```

4