2, 2, 4, 4, 6 = 18 total

MATH 136 — Problem Set 5, part 'B' solutions

7.2/71

$$\int \tan^k x\,dx = \int \tan^{k-2} x \tan^2 x\,dx$$

$$= \int \tan^{k-2} x\,(\sec^2 x - 1)\,dx \quad (1) \qquad \text{(using Hint)}$$

$$= \int \tan^{k-2} x \sec^2 x\,dx - \int \tan^{k-2} x\,dx$$

u-sub: $u = \tan x \quad du = \sec^2 x\,dx$

$$= \frac{1}{k-1}\tan^{k-1} x - \int \tan^{k-2} x\,dx \quad (1)$$

7.2/78 using the identity $\sin(mx)\cos(nx) = \frac{1}{2}(\sin(m-n)x + \sin(m+n)x)$

$$\int \sin(mx)\cos(nx)\,dx = \int \frac{1}{2}(\sin(m-n)x + \sin(m+n)x)\,dx \quad (1)$$

$$= \frac{1}{2}\int \sin(m-n)x\,dx + \frac{1}{2}\int \sin(m+n)x\,dx$$

$$= \frac{-1}{2(m-n)}\cos(m-n)x + \frac{-1}{2(m+n)}\cos(m+n)x + C \quad (1)$$

which is the given entry in Eq. (18)

7.2/79 $\int \sec^m x\,dx = \int \sec^{m-2} x \sec^2 x\,dx$

Let $u = \sec^{m-2} x \qquad \overbrace{\qquad\qquad}^{(1)} \qquad dv = \sec^2 x\,dx$

$du = (m-2)\sec^{m-3} x \cdot \sec x \tan x\,dx \qquad v = \tan x$

(1)

So $\int \sec^m x\,dx = \sec^{m-2} x \tan x - (m-2)\int \sec^{m-2} x \tan^2 x\,dx$

(1)

$$= \sec^{m-2}x \tan x - (m-2)\int \sec^{m-2}x\,(\sec^2 x - 1)\,dx \quad \text{(identity)}$$

$$= \sec^{m-2}x \tan x - \underbrace{(m-2)\int \sec^m x\,dx}_{\text{add to both sides}} + (m-2)\int \sec^{m-2}x\,dx$$

$$(m-1)\int \sec^m x\,dx = \sec^{m-2}x \tan x + (m-2)\int \sec^{m-2}x\,dx$$

So $$\boxed{\int \sec^m x\,dx = \frac{\sec^{m-2}x \tan x}{m-1} + \frac{m-2}{m-1}\int \sec^{m-2}x\,dx.} \quad 1$$

7.3/47 the average height is

$$\bar{y} = \underbrace{\frac{1}{2}\int_{-1}^{1}\sqrt{1-x^2}\,dx}_{1} \left(= \frac{\pi}{4}\right)$$

(since $\int_{-1}^{1}\sqrt{1-x^2}\,dx$ gives the area of the semicircle.)

Using a trig substitution, $\sqrt{1-x^2}$ means we want $x = \sin\theta$ $dx = \cos\theta\,d\theta$ 1

$$\bar{y} = \frac{1}{2}\int_{\theta=-\pi/2}^{\theta=\pi/2} \cos\theta \cdot \cos\theta\,d\theta$$

$$= \frac{1}{2}\int_{-\pi/2}^{\pi/2} \cos^2\theta\,d\theta \quad 1 \qquad \text{(double-angle)}$$

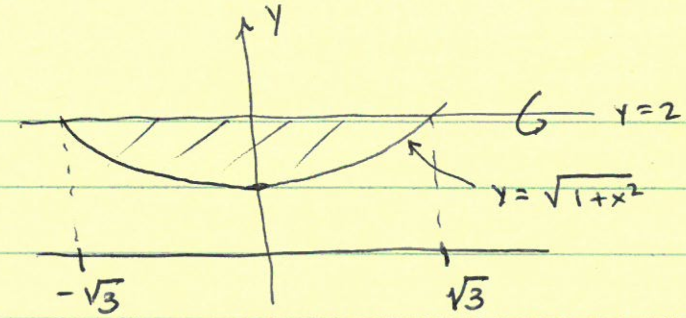
$$= \frac{1}{2}\int_{-\pi/2}^{\pi/2} \frac{1}{2}(1 + \cos 2\theta)\,d\theta$$

$$= \frac{1}{2}\cdot\left[\frac{\theta}{2} + \frac{1}{4}\sin(2\theta)\right]\Bigg|_{-\pi/2}^{\pi/2}$$

$$= \frac{1}{2}\left[\frac{\pi}{4} + \frac{\pi}{4} + 0 - 0\right] = \boxed{\frac{\pi}{4}} \quad 1$$

(Also ok if they convert back to $\frac{1}{4}\sin^{-1}x + \frac{1}{4}x\sqrt{1-x^2}\Big|_{-1}^{1}$)

49.



(part of a hyperbola)

Disk - cross sections:

(It's actually somewhat easier to use symmetry and compute

$$2 \cdot \int_0^{\sqrt{3}} \pi \left(2 - \sqrt{1+x^2}\right)^2 dx.)$$

that way is fine too!

$$V = \int_{-\sqrt{3}}^{\sqrt{3}} \pi \left(2 - \sqrt{1+x^2}\right)^2 dx$$

$$= \int_{-\sqrt{3}}^{\sqrt{3}} \pi \left(4 - 4\sqrt{1+x^2} + 1 + x^2\right) dx$$

$$= \pi \int_{-\sqrt{3}}^{\sqrt{3}} 5 + x^2 \, dx - 4\pi \int_{-\sqrt{3}}^{\sqrt{3}} \sqrt{1+x^2} \, dx \qquad \begin{aligned} x &= \tan\theta \\ dx &= \sec^2\theta \, d\theta \end{aligned}$$

$$= \pi \left[5x + \frac{x^3}{3}\right]\Bigg|_{-\sqrt{3}}^{\sqrt{3}} - 4\pi \int_{-\pi/3}^{\pi/3} \sec^3\theta \, d\theta$$

) done in class

$$= \pi\left[10\sqrt{3} + 2\sqrt{3}\right] - 4\pi \left[\frac{1}{2}\sec\theta\tan\theta + \frac{1}{2}\ln|\sec\theta + \tan\theta|\right]\Bigg|_{-\pi/3}^{\pi/3}$$

$$= \pi \cdot (12\sqrt{3}) - 2\pi\left(4\sqrt{3} + \ln|2+\sqrt{3}| - \ln|2-\sqrt{3}|\right)$$

$$= 4\pi\sqrt{3} - 2\pi \ln\left|\frac{2+\sqrt{3}}{2-\sqrt{3}}\right|$$

$$= 4\pi\sqrt{3} - 2\pi \ln\left|\frac{2+\sqrt{3}}{2-\sqrt{3}} \cdot \frac{2+\sqrt{3}}{2+\sqrt{3}}\right|$$

$$= 4\pi\sqrt{3} - 4\pi \ln(2+\sqrt{3})$$

$$= 4\pi\left(\sqrt{3} - \ln(2+\sqrt{3})\right)$$

(as given in answer.)