Mathematics 136 – Calculus 2
Exam 3 – Solutions for Review Sheet Sample Problems
November 18, 2016

Review problems

There is a selection of review problems posted on WebAssign as an optional assignment. I have increased the number of tries on each problem to the maximum possible – 100.

Sample problems

Note: The actual exam will be considerably shorter than the following list of questions and it might not contain questions of all of these types. The purpose here is just to give an idea of the range of different topics that will be covered and how questions might be posed.

I. *Partial Fractions.*

(A) Do you need partial fractions to integrate

$$\int \frac{t^2+1}{t^3+3t+3}\,dt?$$

Explain, and compute the integral with a simpler method.

Solution: You don't need partial fractions because the substitution $u = t^3 + 3t + 3$ gives $du = 3t^2 + 3$ and the integral is just

$$\frac{1}{3}\int \frac{1}{u}\,du = \frac{1}{3}\ln|u| + C = \frac{1}{3}\ln|t^3+3t+3| + C$$

(B) Apply partial fraction decomposition to compute

$$\int \frac{x^3+1}{x(x-1)(x+2)}\,dx$$

(Hint: you need to divide through first; do you see why?)

Solution: You need to divide first because the degree of the numerator is the same as the degree of the denominator. The result is

$$\frac{x^3}{x^3+x^2-2x} = 1 + \frac{-x^2+2x+1}{x(x-1)(x+2)}$$

We do partial fractions on the second term:

$$\frac{-x^2+2x+1}{x(x-1)(x+2)} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x+2}$$

Clear denominators to obtain:

$$-x^2+2x+1 = A(x-1)(x+2) + Bx(x+2) + Cx(x-1)$$

Equating coefficients of $x^2, x, 1$, this gives

$$\begin{aligned} -1 &= A+B+C \\ 2 &= A+2B-C \\ 1 &= -2A. \end{aligned}$$

From the last of these $A = -\frac{1}{2}$ and then

$$\begin{aligned} -\frac{1}{2} &= B+C \\ \frac{5}{2} &= 2B-C \end{aligned}$$

Adding these gives $B = \frac{2}{3}$ and then $C = -\frac{7}{6}$. The integral is

$$\int 1 + \frac{-1/2}{x} + \frac{2/3}{x-1} - \frac{7/6}{x+2}\, dx = x - \frac{1}{2}\ln|x| + \frac{2}{3}\ln|x-1| - \frac{7}{6}\ln|x+2| + C.$$

(C) What would the partial fractions look like to integrate

$$\int \frac{1}{x(x^2+4)^2}\, dx?$$

(You don't need to solve for the coefficients, just set up the proper partial fractions.)

Solution: The partial fractions have the form

$$\frac{A}{x} + \frac{Bx+C}{x^2+4} + \frac{Dx+E}{(x^2+4)^2}.$$

(D) Suppose the partial fraction decomposition of a rational function is

$$\frac{1}{81x} + \frac{-\frac{x}{9}+1}{(x^2+9)^2} - \frac{\frac{x}{81}}{x^2+9}$$

What is the integral? (Hint: There's one part of this for which you need a trigonometric substitution.)

We split the middle term into two parts, one with $-\frac{x}{9}$ in the numerator, one with 1 in the numerator. The parts which can be integrated immediately are

$$\int \frac{1}{81x} + \frac{-\frac{x}{9}}{(x^2+9)^2} - \frac{\frac{x}{81}}{x^2+9}\, dx = \frac{1}{81}\ln|x| + \frac{1}{18(x^2+9)} - \frac{1}{162}\ln|x^2+9| \qquad (1)$$

(the form of the middle term here is $\int u^{-2}\, du$; the last one is $\int u^{-1}\, du$, both up to suitable constants). The remaining part to be done is

$$\int \frac{1}{(x^2+9)^2}\, dx$$

For this one, we can use the trig substitution $x = 3\tan\theta$, $dx = 3\sec^2\theta d\theta$. Then

$$\int \frac{1}{(x^2+9)^2}\, dx = \frac{1}{27}\int \frac{\sec^2\theta}{\sec^4\theta} d\theta = \frac{1}{27}\int \cos^2\theta\ d\theta.$$

This is integrated by the double angle formula as usual:

$$= \frac{1}{27}\left(\frac{\theta}{2} + \frac{1}{2}\sin\theta\cos\theta\right) + C$$

Finally converting back to a function of x, we have $\tan\theta = x/3$, so $\sin\theta = \frac{x}{\sqrt{x^2+9}}$ and $\cos\theta = \frac{3}{\sqrt{x^2+9}}$. So this remaining part of the integral is

$$= \frac{1}{54}\tan^{-1}\left(\frac{x}{3}\right) + \frac{1}{18}\left(\frac{x}{x^2+9}\right). \qquad (2)$$

The final answer is the sum of (1) and (2).

II. For each of the following integrals, say why the integral is improper, determine if the integral converges, and if so, find its value.

A) $\displaystyle\int_1^\infty \frac{1}{\sqrt[5]{x}}\, dx$

Solution: This is improper because of the infinite interval.

$$\lim_{b\to\infty}\int_1^b x^{-1/5}\, dx = \lim_{b\to\infty}\frac{5}{4}x^{4/5}\Big|_1^b = \lim_{b\to\infty}\frac{5}{4}\left(b^{4/5}-1\right).$$

This is not finite, so the integral *diverges – it does not converge.*

B) $\displaystyle\int_0^2 \frac{dx}{x^2-7x+6}$

Solution: This one is improper because the denominator $x^2-7x+6 = (x-1)(x-6)$ is zero at $x = 1$, which is in the interior of the interval. In order for the integral to converge, *both*

$$\lim_{b\to 1^-}\int_0^b \frac{dx}{x^2-7x+6}$$

and

$$\lim_{a\to 1^+}\int_a^2 \frac{dx}{x^2-7x+6}$$

must exist. However neither integral does exist. We integrate the first by partial fractions and find

$$\begin{aligned}\lim_{b\to 1^-}\int_0^b \frac{dx}{x^2-7x+6} &= \lim_{b\to 1^-}\left(-\frac{1}{5}\ln|x-1| + \frac{1}{5}\ln|x-6|\Big|_0^b\right)\\ &= \lim_{b\to 1^-}\left(-\frac{1}{5}\ln|b-1| + \frac{1}{5}(\ln|b-6| - \ln(6))\right).\end{aligned}$$

But the first term here does not have a finite limit as $b \to 1^-$. Therefore this integral also *diverges.*

C) $\int_0^\infty xe^{-3x}\,dx$

Solution: We integrate by parts with $u = x$, $dv = e^{-3x}\,dx$ and find

$$\begin{aligned} &= \lim_{b\to\infty} -\frac{xe^{-3x}}{3} - \frac{e^{-3x}}{9}\Big|_0^b \\ &= \lim_{b\to\infty} -\frac{b}{3e^{3b}} - \frac{1}{9e^{3b}} + \frac{1}{9} \\ &= \frac{1}{9} \end{aligned}$$

using L'Hopital's Rule on the first term (the limit of each of the first two terms is zero). This integral *converges*

D) For which values of a is $\int_0^\infty e^{ax}\sin(x)\,dx$ convergent? Evaluate the integral for those a.

Solution: The only chance here is if $a < 0$ so the exponential is decaying as $x \to +\infty$. If this is true then we integrate by parts twice to get

$$\int e^{ax}\sin(x)\,dx = \frac{e^{ax}}{1+a^2}(a\sin(x) - \cos(x))$$

since $a < 0$, the exponential goes to zero as $x \to \infty$ and the value of the integral is just the negative of the value at $x = 0$:

$$\int_0^a e^{ax}\sin(x)\,dx = \frac{1}{1+a^2}.$$

III. The time t (in minutes) between two successive incoming calls at a phone center is a random variable with a pdf of the form $f(t) = \frac{1}{r}e^{-t/r}$ if $t \geq 0$ and 0 otherwise.

(A) Suppose that the probability that two successive calls are more than one minute apart is .3. What is the value of r?

Solution: We have

$$.3 = \int_1^\infty \frac{1}{r}e^{-t/r}\,dt = e^{-1/r}$$

So then $-1/r = \ln(.3)$, and $r = -1/\ln(.3) \doteq .8306$.

(B) Using the value of r you determined in part A, find the probability that the two successive calls are at least 5 minutes apart.

Solution: This probability is

$$P(T > 5) \doteq \int_5^\infty \frac{1}{.8306}e^{-t/.8306}\,dt \doteq .0024$$

(that is about a .2% chance of this happening).

IV. *Numerical Integration.*

(A) If you apply the Trapezoidal Rule and the Midpoint Riemann Sum Rule with $N = 30$ to approximate $\int_1^8 x^2 e^x \, dx$, one will overestimate the actual value and one will underestimate the actual value. Which is which? Also, how would we expect the absolute values of the errors for these methods to compare?

Solution: Because $y'' = (x^2+4x+2)e^x > 0$ for all $x > 0$, the graph $y = x^2 e^x$ is concave up for all $x > 0$. This means that the Trapezoidal Rule will give an *overestimate* and the Midpoint Rule will give an *undererestimate.*

(B) How big would you need to take N to get a Midpoint Riemann Sum approximation to $\int_0^1 \sin(x)\, dx$ accurate to 5 decimal places. (Hint: You want the error bound $< .000005$.)

Solution: The second derivative of $\sin(x)$ is $-\sin(x)$ again. So a cheap estimate for the K_2 in the Midpoint Rule error bound is

$$K_2 = 1 \geq |-\sin(x)|$$

for all x. Then

$$\frac{(1-0)^3}{24N^2} < .000005 \Leftrightarrow N > \sqrt{\frac{1}{24 \cdot .000005}} = 91.29.$$

So any $N \geq 92$ will be sufficient. (Actually somewhat smaller N would work too, since $|-\sin(x)|$ does not get as large as 1 on the interval [0,1]. A smaller value of K_2 could also be used: $K_2 = \sin(1) \doteq .8415$, and that would reduce N accordingly.)

V. *Differential Equations.*

(A) Verify that for any constant C, the function $y = \sqrt{C + x^2}$ is a solution of the differential equation $y\dfrac{dy}{dx} - x = 0$. Which of these solutions also satisfies the initial condition $y(0) = 5$?

Solution: Computing the derivative with the Chain Rule:

$$y\frac{dy}{dx} = \sqrt{C+x^2} \cdot \frac{2x}{2\sqrt{C+x^2}} = x$$

Therefore $y = \sqrt{C+x^2}$ is a solution.

(B) For which real a is $y = \sin(ax)$ a solution of the differential equation $\frac{d^4y}{dx^4} - 16y = 0$?

Solution: if $y = \sin(ax)$, then $\frac{d^4y}{dx^4} = a^4 \sin(ax)$. So $\frac{d^4y}{dx^4} - 16y = 0$ if and only if $a^4 = 16$, so $a = \pm 2$.

(C) Find the general solution of

$$\sqrt{1-x^2}\frac{dy}{dx} = y^2 + 2y$$

Solution: This is a separable equation:

$$\frac{dy}{y(y+2)} = \frac{1}{\sqrt{1-x^2}}\ dx$$

Integrate with partial fractions on the left:

$$\frac{1}{y(y+2)} = \frac{A}{y} + \frac{B}{y+2}$$

when $A = \frac{1}{2}$ and $B = -\frac{1}{2}$. So

$$\begin{aligned}
\int \frac{1/2}{y} - \frac{1/2}{y+2}\ dy &= \int \frac{1}{\sqrt{1-x^2}}\ dx \\
\frac{1}{2}\ln|y| - \frac{1}{2}\ln|y+2| &= \sin^{-1}(x) + C \\
\ln\left|\frac{y}{y+2}\right| &= 2\sin^{-1}(x) + C' \qquad (C' = 2C) \\
y &= \frac{2ae^{2\sin^{-1}(x)}}{1 - ae^{2\sin^{-1}(x)}}.
\end{aligned}$$

where $a = \pm e^{C'}$.

(C) Find the general solution of

$$\frac{dy}{dx} = xe^{x+y}$$

Solution: This is also separable:

$$\int e^{-y}\ dy = \int xe^x\ dx.$$

On the right integrate by parts with $u = x$ $dv = e^x\ dx$. So

$$-e^{-y} = xe^x - e^x + C$$

Then

$$y = -\ln(-xe^x + e^x + C)$$

(I have written the arbitrary constant as C again; in the final version, the constant is the negative of the one in the line before.)

(D) A baked potato is taken out of the oven at a temperature of 140° C and left to cool on a counter in a room maintained at 20° C. After 2 minutes the temperature of the potato has decreased to 100° C. How long will it take for the temperature to reach 80° C?

Solution: The relevant differential equation is the one from Newton's Law of Cooling. The initial condition comes from thinking of the time when the potato comes out of the oven as $t = 0$. Letting T be the temperature of the potato in degrees C, t be time in minutes:

$$\begin{aligned} \frac{dT}{dt} &= k(T-20) \\ T(0) &= 140 \end{aligned}$$

The solution is

$$T(t) = 20 + 120e^{kt}$$

(with $k < 0$). We are given $T(2) = 100$. So

$$100 = 20 + 120e^{2k}$$

and $k = \frac{1}{2}\ln(2/3) \doteq -.20273$. Then we want to solve for t in the equation from the time when $T = 80$:

$$80 = 20 + 120e^{(-.20273)t}$$

which shows

$$t = \frac{\ln(1/2)}{-.20273} \doteq 3.42$$

It will take a total of 3.42 minutes (equivalently, another 1.42 minutes after the time when $T = 100$).

(E) Sunset Lake is stocked with 2000 rainbow trout. After 2 years the trout population has grown to 4500. Assuming logistic growth with a carrying capacity of 10000 trout, find the growth constant k and determine when the population will reach 5000.

Solution: Let y be the trout population as a function of time t in years. We are given that the growth is logistic, which means that

$$\frac{dy}{dt} = ky\left(1 - \frac{y}{10000}\right)$$

for some k. We also know the general solution will look like this:

$$y = \frac{10000}{1 + ae^{-kt}}$$

Take $t = 0$ to be the time when the lake is stocked, so $y(0) = 2000$. (We need to assume there were no trout there before.) Then

$$2000 = \frac{10000}{1+a} \Rightarrow a = 4.$$

Next at $t = 2$ we have

$$\begin{aligned} 4500 &= \frac{10000}{1+4e^{-2k}} \\ 1 + 4e^{-2k} &= \frac{10000}{4500} \doteq 2.222 \\ k &\doteq -\frac{1}{2}\ln(1.222/4) \doteq .5929 \end{aligned}$$

The population will reach 5000 when

$$5000 = \frac{10000}{1 + 4e^{-(.5929)t}}.$$

Solving for t:

$$t = \frac{\ln(1/4)}{-.5929} \doteq 2.3$$

years. That is, in another 0.3 years after $t = 2$.

(F) Find the general solution of

$$\frac{dy}{dx} + xy = x$$

Solution: This one is first order linear. The integrating factor is $e^{\int x\ dx} = e^{x^2/2}$. Then the general solution is

$$y = Ce^{-x^2/2} + e^{-x^2/2} \int e^{x^2/2} x\ dx$$

By a u-substition, $\int e^{x^2/2} x\ dx = e^{x^2/2}$, so the general solution simplifies to

$$y = Ce^{-x^2/2} + 1.$$