MATH 136 – Calculus 2
Discussion – Archimedes' *Quadrature of the Parabola*
November 22, 2016

Background

One of the most interesting surviving works of the ancient Greek mathematician Archimedes (287 - 212 BCE) is his *Quadrature of the Parabola*. The problem he solved was that of determining the area of the region between a a parabola and one of its secant lines – a region called a *parabolic segment*. The result he showed (with two different proofs, each of an extremely high degree of ingenuity) was:

The area of the parabolic segment bounded the secant line AB is $\frac{4}{3}$ of the area of the triangle ΔABC, where C is the point on the parabola whose tangent line is parallel to AB. (See Figure 1 for a special case.) [1]

We will retrace his steps today, using calculus to derive some of the intermediate results he did via pure geometry.

Questions

(A) Say the parabola is $y = x^2$ and the points A, B are $A = (-1, 1)$ and $B = (2, 4)$ respectively (see Figure 1 on the next page). Find the equation of the secant line and compute the area of the region between the secant line and the parabola using calculus.

(B) Find the point C on the parabola where the tangent line is parallel to the secant line through A and B. Determine the area of ΔABC and verify that Archimedes' result holds.

Needless to say, though, what you did is not what Archimedes did. Moreover he demonstrated the statement above for all points A, B, not just the specific ones in parts A,B. His idea was the following: No matter where A and B are, once we have the vertex C, the two secant lines AC and BC form two new parabolic segments, each with a vertex. These points are called C' and C'' in Figure 2. Archimedes' key insight here was that each of the triangles $\Delta ACC'$ and $\Delta BCC''$ had $\frac{1}{8}$ of the area of ΔABC. Therefore

$$\text{area}(\Delta ACC') + \text{area}(\Delta BCC'') = \frac{1}{4} \cdot \text{area}(\Delta ABC). \qquad (1)$$

(C) With the points C', C, C'' we now have 4 new parabolic segments. We repeat the process of finding the vertex and inscribing a triangle in each one. Explain why (1) shows that the area of the four new triangles obtained this way is

$$\frac{1}{4^2} \cdot \text{area}(\Delta ABC) = \frac{1}{16} \cdot \text{area}(\Delta ABC).$$

[1] "Quadrature" for Archimedes meant finding a *square* with the same area as another region under consideration. Other geometric constructions known to the Greeks would allow a geometer to get a square with the same area as 4/3 times the area of the triangle. Areas of other figures with curved boundaries–for instance circles–had been determined before this. This was the first determination of an area quite this complicated, though. The word "quadrature" later evolved into a near synonym for "integral."

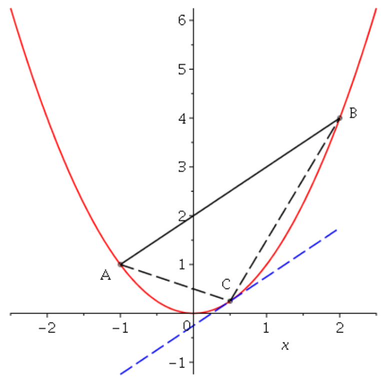

Figure 1: The setting for Archimedes' result

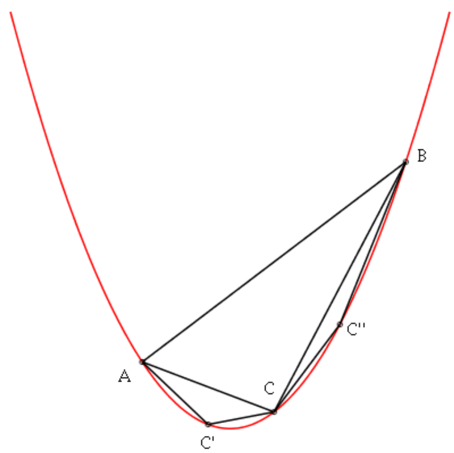

Figure 2: Inscribing new triangles

(D) If we continue creating new triangles this way indefinitely, and add the areas, explain why the total area is given by the following infinite series:

$$\text{area}(\Delta ABC) \cdot \sum_{n=1}^{\infty} \frac{1}{4^n}$$

Determine the sum of this series. On the other hand, it is clear that the triangles we are looking at eventually "fill up" all of the area between the parabola and the secant line AB. This was the way Archimedes proved his result in his *first proof*. The second one is even more amazing!