College of the Holy Cross, Fall 2016
MATH 136, Section 02, Final Exam
Solutions

I.

(A) (5) Compute the derivative of the function $g(x) = \displaystyle\int_0^{3x} \frac{\sin(t)}{t}\, dt$.

Solution: By part 1 of the FTC and the Chain Rule, this is $\frac{\sin(3x)}{3x} \cdot 3 = \frac{\sin(3x)}{x}$.

(B) (10) Let $f(x) = \begin{cases} 6x+2 & \text{if } 0 \le x \le 1 \\ -x+9 & \text{if } 1 < x \le 4 \\ 5 & \text{if } 4 < x \le 5 \end{cases}$ which is plotted in the graph at the top of the next page.

Let $F(x) = \int_0^x f(t)\, dt$, where $f(t)$ is the function above. Complete the following table of values for $F(x)$:

x	0	1	2	3	4	5
$F(x)$	0	5	25/2	19	49/2	59/2

(C) (5) On which interval(s) contained in $(0, 5)$ is the graph $y = F(x)$ concave down?

Solution: Concave down when $F'(x) = f(x)$ is decreasing, so $(1, 4)$.

II.

(A) (5) Use a left-hand Riemann sum with $n = 4$ to approximate $\displaystyle\int_0^1 e^{-x^2/2}\, dx$.

Solution: The value is

$$e^{-0^2/2}(.25) + e^{-(.25)^2/2}(.25) + e^{-(.5)^2/2}(.25) + e^{-(.75)^2/2}(.25) \doteq .9016.$$

(B) (5) Use a midpoint Riemann sum with $n = 4$ to approximate $\displaystyle\int_0^1 e^{-x^2}\, dx$.

Solution: The value is

$$e^{-(.125)^2}(.25) + e^{-(.375)^2}(.25) + e^{-(.625)^2}(.25) + e^{-(.875)^2}(.25) \doteq .8572$$

(C) (10) Check the appropriate boxes and fill in the blank:

The left-hand sum is an overestimate because: $e^{-x^2/2}$ is decreasing on $[0, 1]$. (The first derivative is $-xe^{-x^2/2}$, which is negative for all $x > 0$.)

The midpoint approximation is a overestimate, because $y = e^{-x^2/2}$ is concave down on $[0, 1]$. (The second derivative is $(x^2 - 1)e^{-x^2/2}$, which is negative for all $x \in (-1, 1)$.)

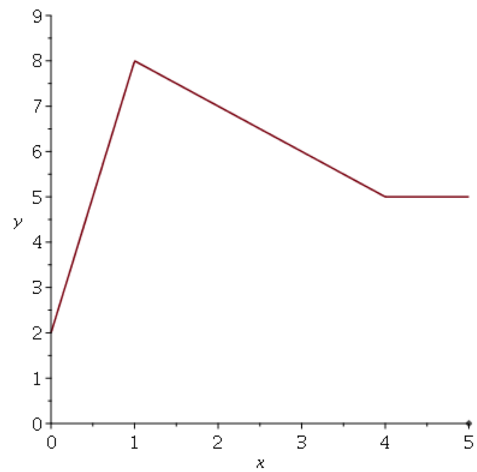

Figure 1: Figure for problem I.

III. Compute the following integrals. Some of these may be forms covered by entries in the table of integrals. Half credit will be given for using a table entry; full credit only for showing all work leading to the final answer.

(A) (5) $\int \frac{x^{2/3} - x^4 + \sqrt{x}}{x^{2/3}}\, dx$

Solution: This equals

$$\int 1 - x^{10/3} + x^{-1/6}\ dx = x - \frac{3}{13}x^{13/3} + \frac{6}{5}x^{5/6} + C.$$

(B) (5) $\int x\cos(x^2)\, dx$

Solution: Let $u = x^2$, then $du = 2x\ dx$, so the integral is

$$\frac{1}{2}\int \cos(u)\ du = \frac{1}{2}\sin(u) + C = \frac{1}{2}\sin(x^2) + C.$$

(C) (10) $\int \frac{\sec^2(5x)\, dx}{(\tan(5x) + 7)^5}$

Solution: Let $u = \tan(5x) + 7$. Then $du = 5\sec^2(5x)$. The integral equals

$$\frac{1}{5}\int u^{-5}\ du = \frac{-1}{20}u^{-4} + C = \frac{-1}{20(\tan(5x) + 7)^4} + C.$$

(D) (10) $\int_1^{e^2} x^3 \ln(x)\, dx.$

Solution: Integrate by parts with $u = \ln(x)$, $dv = x^3\ dx$. Then $du = \frac{1}{x}\ dx$ and $v = \frac{x^4}{4}$, so

$$\begin{aligned}
\int_1^{e^2} x^3 \ln(x)\ dx &= \left.\frac{x^4 \ln(x)}{4}\right|_1^{e^2} - \int_1^{e^2} \frac{x^3}{4}\ dx \\
&= \frac{e^8}{2} - \left.\frac{x^4}{16}\right|_1^{e^2} \\
&= \frac{e^8}{2} - \frac{e^8}{16} + \frac{1}{16} \\
&= \frac{7e^8+1}{16}.
\end{aligned}$$

(E) (15) $\displaystyle\int \frac{x^2}{\sqrt{16-x^2}}\ dx$

Solution: By trigonometric substitution with $x = 4\sin\theta$, so $dx = 4\cos\theta\ d\theta$. We get

$$\int \frac{16\sin^2\theta \cdot 4\cos\theta}{\sqrt{16-16\sin^2\theta}}\ d\theta = 16\int \sin^2\theta\ d\theta.$$

Using the double angle formula $\sin^2\theta = \frac{1}{2}(1-\cos(2\theta))$, this becomes

$$= 8\int 1-\cos(2\theta)\ d\theta = 8\theta - 4\sin(2\theta) + C = 8\theta - 8\sin\theta\cos\theta + C.$$

Then converting back from $x = 4\sin\theta$, $\theta = \sin^{-1}\left(\frac{x}{4}\right)$ and $\cos\theta = \frac{\sqrt{16-x^2}}{4}$. So the final answer is

$$= 8\sin^{-1}\left(\frac{x}{4}\right) - \frac{x\sqrt{16-x^2}}{2} + C.$$

(F) (15) $\displaystyle\int \frac{x+1}{(x-3)(x^2+1)}\ dx$

Solution: By partial fractions:

$$\frac{x+1}{(x-3)(x^2+1)} = \frac{A}{x-3} + \frac{Bx+C}{x^2+1},$$

so

$$x+1 = A(x^2+1) + (Bx+C)(x-3) = (A+B)x^2 + (C-3B)x + A - 3C$$

Hence

$$A+B=0, -3B+C=1, A-3C=1$$

From the second equation $C = 1+3B$, and the last equation becomes $A - 9B = 4$. Solving that with the first equation yields $A = \frac{2}{5}$, $B = \frac{-2}{5}$ and $C = \frac{-1}{5}$. Then

$$\int \frac{x+1}{(x-3)(x^2+1)}\ dx = \int \frac{2/5}{x-3}\ dx + \int \frac{(-2/5)x}{x^2+1}\ dx + \int \frac{-1/5}{x^2+1}\ dx.$$

The final answer is

$$\frac{2}{5}\ln|x-3| - \frac{1}{5}\ln(x^2+1) - \frac{1}{5}\tan^{-1}(x) + C.$$

IV. For each of the following improper integrals, set up and evaluate the appropriate limits to determine whether the integral converges. If so, find its value; if not, say "does not converge."

(A) (10) $\displaystyle\int_1^3 \frac{1}{\sqrt[3]{x-1}}\,dx.$

Solution: Here the discontinuity is at the left endpoint of the interval, so we want

$$\begin{aligned}\lim_{a\to 1^+}\int_a^3 (x-1)^{-1/3}\,dx &= \lim_{a\to 1^+}\frac{3}{2}(x-1)^{2/3}\Big|_a^3 \\ &= \lim_{a\to 1^+}\frac{3}{2}(2^{2/3}-(a-1)^{2/3}) \\ &= \frac{3}{2}2^{2/3}.\end{aligned}$$

Therefore the integral converges to $\frac{3}{2}2^{2/3}$.

(B) (10) $\displaystyle\int_0^\infty \frac{1}{x^2+4x+5}\,dx.$ (Hint: There are two ways to do this. You can compute the integral if you complete the square. Or you can use $\frac{1}{x^2+4x+5} < \frac{1}{x^2}$ for all $x > 0$..)

Solution: This integral is improper because of the infinite limit of integration. Completing the square, we get

$$\begin{aligned}\lim_{b\to\infty}\int_0^b \frac{1}{(x+2)^2+1}\,dx &= \lim_{b\to\infty}\tan^{-1}(x+2)\Big|_0^b \\ &= \lim_{b\to\infty}\tan^{-1}(b+2)-\tan^{-1}(2) \\ &= \frac{\pi}{2}-\tan^{-1}(2).\end{aligned}$$

This integral also converges, to $\frac{\pi}{2}-\tan^{-1}(2) \doteq .4636$.

V. A region R in the plane is bounded by the graphs $y = x^2$, $y = x+6$. See Figure 2.

(A) (10) Compute the area of the region R.

Solution: The parabola crosses the line where $x^2 - x - 6 = (x-3)(x+2) = 0$, so $x = -2, 3$. The area is

$$\begin{aligned}\int_{-2}^3 x+6-x^2\,dx &= \frac{x^2}{2}+6x-\frac{x^3}{3}\Big|_{-2}^3 \\ &= \frac{9}{2}+18-9-2+12-\frac{8}{3} \\ &= \frac{125}{6}.\end{aligned}$$

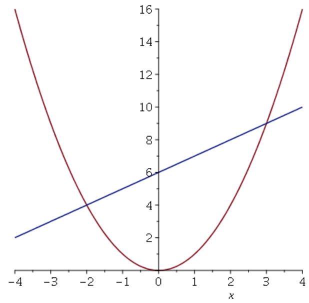

Figure 2: Figure for problem V.

(B) (10) Compute the volume of the solid obtained by rotating R about the x-axis.

Solution: The cross-sections by planes perpendicular to the x-axis are washers with inner radius $r_{in} = x^2$ and outer radius $r_{out} = x + 6$. Therefore the integral that computes the volume is

$$
\begin{aligned}
V &= \int_{-2}^{3} \pi(x+6)^2 - \pi(x^2)^2 \, dx \\
&= \pi \int_{-2}^{3} x^2 + 12x + 36 - x^4 \, dx \\
&= \pi \left. \frac{x^3}{3} + 6x^2 + 36x - \frac{x^5}{5} \right|_{-2}^{3} \\
&= \pi \left(9 + 54 + 108 - \frac{243}{5} + \frac{8}{3} - 24 + 72 - \frac{32}{5} \right) \\
&= \frac{500\pi}{3}.
\end{aligned}
$$

(C) (10) Set up the integral(s) to compute the volume of the solid obtained by rotating R about the line $y = 12$. *You do not need to compute the value.*

Solution: Also washer cross-sections. The integral is

$$V = \int_{-2}^{3} \pi(12 - x^2)^2 - \pi(6 - x)^2 \, dx.$$

VI.

(A) (10) A drug is administered to a patient intravenously at a constant rate of 10mg per hour. The patient's body breaks down the drug and removes it from the bloodstream at a rate proportional to the amount present. Write a differential equation for the function $y(t)$ = amount of the drug present (in mg) in the bloodstream at time t (in hours) that describes this situation. You *do not need to solve* the equation.

Solution: The equation is $\dfrac{dy}{dt} = 10 - ky$. The idea is similar to the derivation of the mixing problem differential equation we did in class. The $\frac{dy}{dt}$ is the rate of change of the amount of the drug, the 10 represents the inflow of the drug into the blood stream via the IV, and the $-ky$ represents the drug being removed at a rate proportional to the amount. I wrote the constant of proportionality here as $-k$ to emphasize that this is the term representing the outflow. The value of k itself would be > 0.

(B) (10) Find the general solution of the differential equation $\frac{dy}{dx} = xy\sqrt{x^2+1}$.

Solution: This is a separable equation. After separating variables

$$\int \frac{dy}{y} = \int x(x^2+1)^{1/2}\,dx.$$

So

$$\begin{aligned} \ln|y| &= \frac{1}{3}(x^2+1)^{3/2} + C \\ y &= De^{\frac{1}{3}(x^2+1)^{3/2}}, \end{aligned}$$

where the arbitrary constant $D = \pm e^C$

(C) (10) Let $y(t)$ represent the population of a colony of tree frogs that is undergoing logistic growth following the differential equation $\frac{dy}{dt} = (.1)y(1 - \frac{y}{100})$, t in years. If the initial population is $y(0) = 10$, how long does it take for the population to reach 45?

Solution: The general solution of this logistic equation is

$$y = \frac{100}{1 + de^{-(0.1)t}}$$

From the initial condition

$$10 = \frac{100}{1+d} \Rightarrow d = 9$$

Then we solve for t:

$$45 = \frac{100}{1 + 9e^{-(0.1)t}}$$

which gives

$$t = -10\ln(11/81) \doteq 19.96$$

or about 20 years.

VII. For full credit, you must justify your answer completely by showing how the indicated test applies and leads to your stated conclusion.

(A) (5) Use the Integral Test to determine if $\displaystyle\sum_{n=1}^{\infty} \frac{n}{e^{2n}}$ converges.

Solution: We need to decide whether $\int_1^\infty xe^{-2x}\,dx$ converges. Integrating by parts with $u = x$, $dv = e^{-2x}\,dx$, we have

$$\begin{aligned} \lim_{b\to\infty} -\frac{x}{2e^{2x}} - \frac{1}{4e^{2x}}\Big|_1^b &= \lim_{b\to\infty}\left(-\frac{b}{2e^{2b}} - \frac{1}{4e^{-2b}} + \frac{3}{4e^2}\right) \\ &= \frac{3}{4e^2}. \end{aligned}$$

(We used L'Hopital's Rule here to see $\lim_{b\to\infty} \frac{b}{2e^{2b}} = 0$.) Since the improper integral converges, the series does too.

(B) (15) Using the Ratio Test and testing convergence at the endpoints, determine the interval of convergence for the power series

$$\sum_{n=1}^{\infty} \frac{x^n}{n3^n}.$$

Solution: The Ratio Test gives

$$\lim_{n\to\infty}\left|\frac{x^{n+1}}{(n+1)3^{n+1}} \cdot \frac{n3^n}{x^n}\right| = \lim_{n\to\infty} \frac{1}{3}\frac{n}{n+1}|x| = \frac{1}{3}|x|.$$

This says the series converges absolutely when $|x|/3 < 1$, or $|x| < 3$. It diverges if $|x| = 3$. Substituting $x = 3$ into the series we have $\sum_{n=1}^{\infty} \frac{1}{n}$, the divergent harmonic series. On the other hand $x = -3$ gives $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$, which converges by the Alternating Series Test. Thus the full interval of convergence is $[-3, 3)$, or $-3 \le x < 3$.

Extra Credit. (10) In economics, the *multiplier effect* refers to the fact that when there is an injection of money to consumers in an economy, the consumers spend a certain proportion of it. That amount recirculates through the enconomy and adds additional income, which comes back to the consumers and they spend the same percentage. The process repeats *indefinitely* circulating additional money through the economy. Suppose that in order to stimulate the economy, the government cuts taxes by \$ 50 billion, thereby injecting that much money back to consumers. If consumers save 10% of the money they get and spend the other 90%, what is the total additional spending circulated through the economy by the tax cut?

Solution: The total is the sum of the geometric series

$$50 + 50(.9) + 50(.9)^2 + 50(.9)^3 + \cdots = \frac{50}{1 - .9} = 500$$

(units are billions of dollars).