College of the Holy Cross, Fall 2016
Math 136, section 2 – Solutions for Midterm Exam 2 Retest
Tuesday, November 8

I.

A. (15) Integrate by parts: $\int \tan^{-1}(x)\ dx$. (Note there's really only one choice for u and hence dv.)

Solution: The only possible choice that leads to anything of value is $u = \tan^{-1}(x)$ and $dv = dx$, so $du = \frac{1}{x^2+1}\ dx$ and $v = x$. Then

$$\int \tan^{-1}(x)\ dx = x\tan^{-1}(x) - \int \frac{x}{x^2+1}\ dx.$$

This remaining integral can be done via the substitution $w = x^2 + 1$, which takes it to $-\frac{1}{2}\int \frac{dw}{w} = -\frac{1}{2}\ln|x^2+1|$. The full answer is

$$x\tan^{-1}(x) - \frac{1}{2}\ln|x^2+1| + C.$$

B. (15) Integrate using appropriate trigonometric identities: $\int \sin^3(6x)\cos^4(6x)\ dx$.

Solution: With the odd power of sine, we split off one factor of $\sin(6x)$ and use $\sin^2(6x) = 1 - \cos^2(6x)$ to convert the rest:

$$\begin{aligned}
\int \sin^3(6x)\cos^4(6x)\ dx &= \int \sin(6x)(1-\cos^2(6x))\cos^4(6x)\ dx \\
&= \int \cos^4(6x)\sin(6x)\ dx - \int \cos^6(6x)\sin(6x)\ dx \qquad (u = \cos(6x)) \\
&= -\frac{1}{6}\int u^4\ du + \frac{1}{6}\int u^6\ du \\
&= -\frac{1}{30}u^5 + \frac{1}{42}u^7 + C \\
&= -\frac{1}{30}\cos^5(6x) + \frac{1}{42}\cos^7(6x) + C.
\end{aligned}$$

C. (20) Integrate with a trigonometric substitution: $\int \sqrt{25 - x^2}\ dx$. (Partial credit points will be given as follows: 5 points for the substitution equation $x = \ldots$, and the computation of dx, 5 points for the conversion to the trigonometric integral, 5 points for the integration of the trigonometric integral, 5 points for conversion back to the equivalent function of x.)

Solution: Let $x = 5\sin(\theta)$, then $dx = 5\cos(\theta)\ d\theta$. The integral substitutes to

$$\int \sqrt{25 - 25\sin^2(\theta)} \cdot 5\cos(\theta)\ d\theta = 25 \int \cos^2(\theta)\ d\theta.$$

For this we use the identity $\cos^2(\theta) = \frac{1}{2}(1 + \cos(2\theta))$, so the integral becomes

$$25 \int \frac{1}{2}(1 + \cos(2\theta))\ d\theta = \frac{25\theta}{2} + \frac{25}{4}\sin(2\theta) + C,$$

which can also be rewritten using the double angle formula for the sine:

$$\frac{25\theta}{2} + \frac{25}{2}\sin(\theta)\cos(\theta) + C.$$

Converting back to functions of x, we have $\frac{x}{5} = \sin(\theta)$, so $\theta = \sin^{-1}\left(\frac{x}{5}\right)$ and $\cos(\theta) = \frac{\sqrt{25-x^2}}{5}$. So after simplifying we get

$$\frac{25}{2}\sin^{-1}\left(\frac{x}{5}\right) + \frac{x}{2}\sqrt{25 - x^2} + C.$$

D. (5) Derive the reduction formula (assume $m \geq 2$ – this one is *not* done by parts):

$$\int \tan^m(x)\ dx = \frac{\tan^{m-1}(x)}{m-1} - \int \tan^{m-2}(x)\ dx.$$

Solution: (This same problem appeared on Problem Set 5 part B.) Break off 2 powers of $\tan(x)$ and convert that to $\sec^2(x) - 1$:

$$\int \tan^m(x)\ dx = \int \tan^{m-2}(x)(\sec^2(x) - 1)\ dx = \int \tan^{m-2}(x)\sec^2(x)\ dx - \int \tan^{m-2}(x)\ dx.$$

The first integral is $\int u^{m-2} du$ for $u = \tan(x)$ so we get

$$= \frac{\tan^{m-1}(x)}{m-1} - \int \tan^{m-2}(x)\ dx,$$

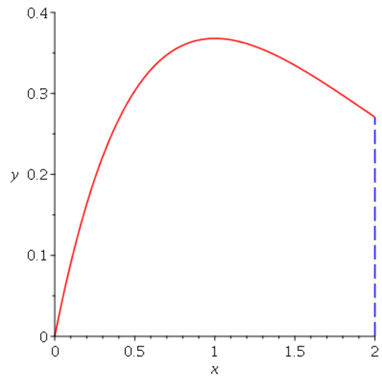
which is what we wanted to show.

E. (5) Use the formula from part D repeatedly to integrate $\displaystyle\int \tan^4(x)\ dx$

Solution: Applying the formula from part D twice, with $m = 4$, then $m = 2$, we get:

$$\begin{aligned} \int \tan^4(x)\ dx &= \frac{\tan^3(x)}{3} - \int \tan^2(x)\ dx \\ &= \frac{\tan^3(x)}{3} - \tan(x) + \int 1\ dx \\ &= \frac{\tan^3(x)}{3} - \tan(x) + x + C \end{aligned}$$

II. All parts of this question refer to the region R bounded by $y = xe^{-x}$ (in red), the x-axis, $x = 0$ and $x = 2$.



A. (10) Set up and compute integral(s) to find the area of R.

Solution: The area is

$$A = \int_0^2 xe^{-x}\, dx.$$

Integrating by parts $(u = x, dv = e^{-x})$ we get

$$-xe^{-x} - e^{-x}\Big|_0^2 = 1 - 3e^{-2} \doteq .594$$

B. (10) The region R is rotated about the x-axis to generate a solid. Set up and compute an integral to find its volume.

Solution: The volume has disk cross-sections so

$$V = \int_0^2 \pi(xe^{-x})^2\, dx = \pi \int_0^2 x^2 e^{-2x}\, dx$$

So we integrate by parts again (*twice* this time with $u = x^2$, then $u = x$). The final answer is

$$V = \frac{\pi}{4}(1 - 13e^{-4}).$$

C. (10) A solid has the portion of the region R as base. The cross-sections by planes perpendicular to the x-axis are equilateral triangles extending from the x-axis up to the point on the graph $y = xe^{-x}$. *Set up* an integral to find the volume. You do not need to compute this one, but if you do correctly, I will give 10 points Extra Credit.

Solution: The area of an equilateral triangle of side s is $\frac{s^2\sqrt{3}}{4}$, so the integral is

$$\int_0^2 \frac{\sqrt{3}}{4}(xe^{-x})^2\, dx.$$

If you're clever, you'll note that this is exactly the same integral as in part C, up to the constant factor $\frac{\sqrt{3}}{4\pi}$. So this volume is equal to

$$\frac{\sqrt{3}}{16}(1 - 13e^{-4})$$

with no extra computation(!)

III. (10) The area of a region in the plane is equally distributed about a point called its *centroid*. For a region R bounded by $y = f(x) \geq 0$, the x-axis, $x = a$ and $x = b$, take it as known that the y-coordinate of the centroid is computed by this ratio of two integrals:

$$\overline{y} = \frac{\int_a^b \frac{1}{2}(f(x))^2 \, dx}{\int_a^b f(x) \, dx}.$$

Explain how this result shows that *The volume of the solid generated by rotating R about the x-axis is equal to the product of the area of R and the distance traveled by the centroid as R moves around the x-axis.*[1]

Solution: The centroid travels around a circle of radius $\overline{y}$, so the distance it travels is the circumference, or $2\pi\overline{y}$. The area of the region is $\int_a^b f(x) \, dx$. Therefore the product of the distance traveled by the centroid and the area is:

$$2\pi\overline{y} \cdot \int_a^b f(x) \, dx$$

We can substitute from the given formula for $\overline{y}$ to get

$$2\pi\overline{y} \cdot \int_a^b f(x) \, dx = 2\pi \frac{\int_a^b \frac{1}{2}(f(x))^2 \, dx}{\int_a^b f(x) \, dx} \cdot \int_a^b f(x) \, dx = \int_a^b \pi(f(x))^2 \, dx.$$

We know this is the volume of the solid of revolution by Cavalieri's Principle.

[1]This is a famous result of the ancient Greek mathematician Pappus of Alexandria who lived ca. 290 - 350 CE. He proved it without integral calculus, which had not yet been invented!