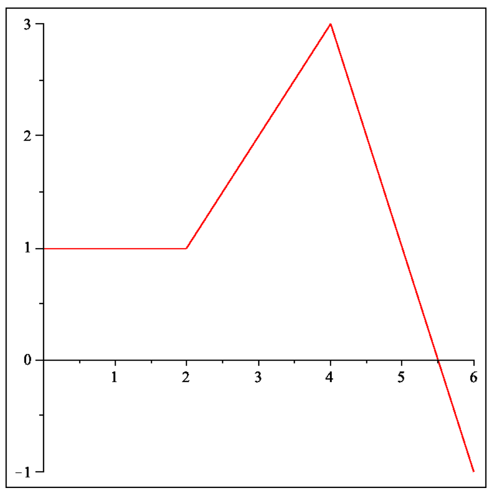
College of the Holy Cross, Fall Semester, 2016
MATH 136, Section 02, Solutions for Practice Final Exam
December 5, 2016

I.

(A) Let $f(x) = \begin{cases} 1 & \text{if } 0 \le x \le 2 \\ x - 1 & \text{if } 2 < x \le 4. \\ 11 - 2x & \text{if } 4 < x \le 6 \end{cases}$ whose graph is shown here:

Solution:



Let $F(x) = \int_0^x f(t)\,dt$, where $f(t)$ is the function from part (B). Complete the following table of values for $F(x)$:

Solution: The value $F(x)$ represents the signed area between the graph $y = f(x)$ and the x-axis:

x	0	1	2	3	4	5	6
$F(x)$	0	1	2	3.5	6	8	8

(B) Compute the derivative of the function $g(x) = \displaystyle\int_0^{2x} \frac{\cos(t)}{t^2}\,dt$.

Solution: By the first part of the Fundamental Theorem of Calculus and the Chain Rule for derivatives:

$$g'(x) = \frac{\cos(2x)}{(2x)^2} \cdot 2 = \frac{\cos(2x)}{2x^2}.$$

II. Compute the following integrals.

(A) $\displaystyle\int \frac{x^4 - 3\pi^3 + \sqrt{x}}{x^{2/3}}\, dx$

Solution: Split into separate fractions, simplify and integrate:

$$\begin{aligned}\int \frac{x^4 - 3\pi^3 + \sqrt{x}}{x^{2/3}}\, dx &= \int x^{10/3} - 3\pi^3 x^{-2/3} + x^{-1/6}\, dx \\ &= \frac{3}{13}x^{13/3} - 9\pi^3 x^{1/3} + \frac{6}{5}x^{5/6} + C.\end{aligned}$$

(B) $\displaystyle\int x^3 e^{x^2}\, dx$ (integrate by parts)

Solution: The correct choice is $u = x^2$ and $dv = xe^{x^2}\, dx$ (note that you need the x with the exponential in order to integrate!) Hence $du = 2x\, dx$ and $v = \frac{1}{2}e^{x^2}$. Then by the parts formula

$$\int x^3 e^{x^2}\, dx = \frac{1}{2}x^2 e^{x^2} - \int xe^{x^2}\, dx = \frac{1}{2}x^2 e^{x^2} - \frac{1}{2}e^{x^2} + C.$$

(C) $\displaystyle\int \frac{\csc^2(5x)\, dx}{\cot(5x) + 7}$

Solution: This one can be handled by the u-substitution $u = \cot(5x) + 7$, for which $du = -5\csc^2(5x)\, dx$ by the Chain Rule. Then

$$\int \frac{\csc^2(5x)\, dx}{\cot(5x) + 7} = \frac{-1}{5}\int u^{-1}\, du = \frac{-1}{5}\ln|u| + C = \frac{-1}{5}\ln|\cot(5x) + 7| + C.$$

(D) $\displaystyle\int_1^e x^5 \ln(x)\, dx.$

Solution: Integrate by parts with $u = \ln(x)$ and $dv = x^5\, dx$. Then $du = \frac{1}{x}\, dx$ and $v = \frac{1}{6}x^6$ and by the integration by parts formula,

$$\int x^5 \ln(x)\, dx = \frac{x^6}{6}\ln(x) - \int \frac{1}{6}x^6 \cdot \frac{1}{x}\, dx = \frac{x^6}{6}\ln(x) - \frac{x^6}{36} + C.$$

For the definite integral, we apply the Fundamental Theorem to get

$$\int_1^e x^5 \ln(x)\, dx = \left.\frac{x^6}{6}\ln(x) - \frac{x^6}{36}\right|_1^e = \frac{e^6}{6}\ln(e) - \frac{e^6}{36} - \frac{1}{6}\ln(1) + \frac{1}{36} = \frac{5e^6 + 1}{36}.$$

(E) $\displaystyle\int \frac{1}{\sqrt{16 + x^2}}\, dx$

Solution: This can be done by the trigonometric substitution $x = 4\tan\theta$, so $dx = 4\sec^2\theta\,d\theta$, and $\sqrt{16+x^2} = \sqrt{16(1+\tan^2\theta)} = 4\sec\theta$. This simplifies to

$$\int \frac{1}{4\sec\theta}\cdot 4\sec^2\theta\,d\theta = \int \sec\theta\,d\theta.$$

Now we use the memorized form of this integral, then convert back to x:

$$\begin{aligned}\int \sec\theta\,d\theta &= \ln|\sec\theta+\tan\theta| + C\\ &= \ln\left|\frac{\sqrt{x^2+16}}{4}+\frac{x}{4}\right| + C.\end{aligned}$$

Using properties of logarithms and incorporating $-\ln(4)$ with the constant, this can also be written in the form:

$$\ln|\sqrt{16+x^2}+x| + C.$$

(F) $\displaystyle\int \frac{x}{(x^2+1)(x+1)}\,dx$

Solution: Using partial fractions, we must solve for A, B, C to make:

$$\frac{1}{(x+1)(x^2+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+1}.$$

Clearing denominators,

$$1 = A(x^2+1) + (Bx+C)(x+1),$$

so equating coefficients, $A+B=0$, $B+C=0$, and $A+C=1$. Solving simultaneously,

$$A = \frac{1}{2}, \qquad B = \frac{-1}{2}, \qquad C = \frac{1}{2}.$$

Then

$$\begin{aligned}\int \frac{1}{(x+1)(x^2+1)}\,dx &= \int \frac{\frac{1}{2}}{x+1} + \frac{\frac{-x}{2}+\frac{1}{2}}{x^2+1}\,dx\\ &= \frac{1}{2}\ln|x+1| + \frac{-1}{4}\ln(x^2+1) + \frac{1}{2}\tan^{-1}x + C.\end{aligned}$$

III.

(A) The integral

$$\int_1^3 \sqrt{1+9x^4}\,dx$$

would compute the length of the curve $y = x^3$ from $x=1$ to $x=3$. Use a midpoint Riemann sum with $n=4$ to approximate its value.

Solution: The midpoint Riemann sum approximation is

$$\int_1^3 \sqrt{1+9x^4}\,dx \doteq \left(\sqrt{1+9(1.25)^4}+\sqrt{1+9(1.75)^4}+\sqrt{1+9(2.25)^4}+\sqrt{1+9(2.75)^4}\right)(.5)$$
$$\doteq 25.98.$$

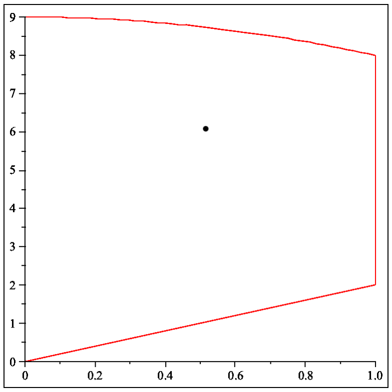
(B) (5)

Solution: The midpoint approximation is an *underestimate* because $\sqrt{1+9x^4}$ is concave up on $[0,2]$.

IV. A region R in the plane is bounded by the graphs $y = 9 - x^2$, $y = 2x$, $x = 0$ and $x = 1$.

(A) Compute the area of the region R.

Solution: Here is a sketch of the region:



The area is given by the integral

$$A = \int_0^1 9 - x^2 - 2x\,dx = 9x - \frac{x^3}{3} - x^2\Big|_0^1 = \frac{23}{3}.$$

(B) Compute the volume of the solid obtained by rotating R about the x-axis.

Solution: The cross-sections of the solid by planes perpendicular to the x-axis are washers with inner radius $r_{in} = 2x$ and outer radius $r_{out} = 9 - x^2$. So the volume is the integral of the area of the cross-section:

$$V = \int_0^1 \pi(9-x^2)^2 - \pi(2x)^2\,dx = \pi\int_0^1 81 - 22x^2 + x^4\,dx$$
$$= \pi\left(81x - \frac{22x^3}{3} + \frac{x^5}{5}\Big|_0^1\right)$$
$$= \frac{1108\pi}{15}.$$

(C) Set up the integral(s) to compute the volume of the solid obtained by rotating R about the y-axis. *You do not need to compute the value.*

Solution: The cross-sections by planes perpendicular to the y-axis are all disks, but the function giving the radius of the disk is given by three different formulas depending on whether $0 \le y \le 2$, or $2 \le y \le 8$, or $8 \le y \le 9$.

$$V = \int_0^2 \pi \left(\frac{y}{2}\right)^2 dy + \int_2^8 \pi(1)^2\, dy + \int_8^9 \pi(\sqrt{9-y})^2\, dy.$$

V. The daily solar radiation x per square meter (in hundreds of calories) in Florida in October has a probability density function $f(x) = c(x-2)(6-x)$ if $2 \le x \le 6$, and zero otherwise. Find value of c and the probability that the daily solar radiation per square meter is greater than 4.

Solution: We must have

$$1 = \int_2^6 c(x-2)(6-x)\, dx = c\left(-\frac{x^3}{3} + 4x^2 - 12x\Big|_2^6\right) = \frac{32c}{3}.$$

Therefore $c = \frac{3}{32}$. Then the probability that $x > 4$ is given by the integral

$$\overline{x} = \int_4^6 \frac{3}{32}(x-2)(6-x)\, dx = \int_4^6 \frac{3x}{4} - \frac{3x^2}{32} - \frac{9}{8}\, dx = \frac{1}{2}.$$

VI. An avian flu epidemic has broken out in Birdsburgh, a large city with total population 10 million. Let N be the number, in millions, of people who have been infected, as a function of time t in weeks. The Birdsburgh Public Health department proposes the model that the rate of change of N is proportional to the product of the number of infected people (N) and the number of people not yet infected.

(A) (10) Write the proposed model above as a differential equation, calling the constant of proportionality k.

Solution: If N people have been infected (N in millions), then the number who have not been infected is $10 - N$ (millions). So the differential equation is

$$\frac{dN}{dt} = kN(10 - N).$$

(Note that this can be put into the form of a logistic equation:

$$\frac{dN}{dt} = (10k)N\left(1 - \frac{N}{10}\right);$$

the 10 plays the role of the carrying capacity.)

(B) (5) The function $N(t) = 10/(1 + 9999e^{-t})$ should be a solution of your differential equation from part A. What is the value of k?

Solution: For this N,

$$\frac{dN}{dt} = \frac{-99990e^{-t}}{(1+9999e^{-t})^2} \tag{1}$$

and

$$\begin{aligned} N(10-N) &= \frac{10}{1+9999e^{-t}} \cdot \left(10 - \frac{10}{1+9999e^{-t}}\right) \\ &= \frac{10 \cdot (-99990e^{-t})}{(1+9999e^{-t})^2} \end{aligned} \tag{2}$$

Hence comparing (1) and (2), we must have $k = \frac{1}{10} = .1$. (Note: This could also be done by recognizing that the differential equation can be rearranged to the form of a logistic equation with constant $10k$: $\frac{dN}{dt} = (10k)N\left(1 - \frac{N}{10}\right)$. The general solution of this is

$$N = \frac{10}{1 + ce^{-(10k)t}}$$

So if the given function is a solution we must have $c = 9999$ and $10k = 1$, so $k = .1$.)

C) If the epidemic proceeds according the function given in part B, how many weeks will pass before the number of infected people reaches 1 million?

Solution: We must solve $1 = \frac{10}{1+9999e^{-t}}$. So $1+9999e^{-t} = 10$, and $t = -\ln(9/9999) = \ln(1111) \doteq 7.0$ weeks. Note that the units of N are millions of people so the left side of the equation is $N = 1$, not $N = 1000000$.

VII.

(A) Does the infinite series $\displaystyle\sum_{n=0}^{\infty} \frac{1}{4^n}$ converge? If so, what is the sum?

Solution: Yes – this is a geometric series with $a = 1$ and $r = \frac{1}{4}$, which is < 1 in absolute value. The sum is $\frac{1}{1-\frac{1}{4}} = \frac{4}{3}$.

(B) Use the Integral Test to determine whether the infinite series $\displaystyle\sum_{n=3}^{\infty} \frac{1}{n(\ln(n))^3}$ converges.

Solution: The improper integral

$$
\begin{aligned}
\int_3^\infty \frac{1}{x(\ln(x))^3}\,dx &= \lim_{b\to\infty}\int_3^b \frac{1}{x(\ln(x))^3}\,dx \\
&= \lim_{b\to\infty} -\frac{1}{2(\ln(x))^2}\Big|_3^b \\
&= \lim_{b\to\infty} \frac{1}{2(\ln(3))^2} - \frac{1}{2(\ln(b))^2}\Big|_3^b \\
&= \frac{1}{2(\ln(3))^2}.
\end{aligned}
$$

Since the integral converges, the series does too.

(C) Use the Ratio Test, then consider the endpoints separately, to determine the interval of convergence of the power series $\sum_{n=0}^{\infty} \frac{x^n}{n^2}$.

Solution: By the Ratio Test:

$$\lim_{n\to\infty}\left|\frac{x^{n+1}}{(n+1)^2}\cdot\frac{n^2}{x^n}\right| = \lim_{n\to\infty}\frac{n^2}{(n+1)^2}|x| = |x|.$$

So the series converges absolutely for $|x| < 1$, or on $(-1, 1)$. At the two endpoints:

$$x = 1 \Rightarrow \sum_{n=1}^{\infty}\frac{1}{n^2}$$

is convergent (a p-series with $p > 1$).

$$x = -1 \Rightarrow \sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}$$

which converges by the Alternating Series Test. Hence the whole interval of convergence is the closed interval $[-1, 1]$.