Mathematics 136 – Calculus 2
Exam 2 – Answers/Solutions for Sample Questions
October 10, 2016

Sample Exam Questions

Disclaimer: The actual exam questions may be organized differently and ask questions in different ways. This list is also quite a bit longer than the actual exam will be (to give you some idea of the range of different questions that might be asked).

I. Compute each of the integrals below using some combination of basic rules, substitution, integration by parts. You must show all work for full credit.

A)

$$\int \frac{e^{\tan^{-1}(x)}}{1+x^2}\,dx$$

Solution: By substitution. Let $u = \tan^{-1}(x)$. The form is $\int e^u\,du$, so the integral is $e^{\tan^{-1}(x)} + C$.

B)

$$\int e^x \sin(2x)\,dx$$

Answer: Apply integration by parts twice; the second time, the original integral is returned, but with a numerical coefficient, so you can solve for it algebraically. The integral equals: $\frac{1}{5}e^x\sin(2x) - \frac{2}{5}e^x\cos(2x) + C$.

C)

$$\int \sin^3(x)\cos^4(x)\,dx$$

Solution: Since the power of $\sin(x)$ is odd, we split off one power of $\sin(x)$, convert the remaining even power to powers of $\cos(x)$, then use the substitution $u = \cos(x)$:

$$\begin{aligned}\int \sin^3(x)\cos^4(x)\,dx &= \int (1-\cos^2(x))\cos^4(x)\sin(x)\,dx \\ &= \int \cos^4(x)\sin(x)\,dx - \int \cos^6(x)\sin(x)\,dx \\ &= \frac{-\cos^5(x)}{5} + \frac{\cos^7(x)}{7} + C.\end{aligned}$$

D)

$$\int_0^1 \tan^{-1}(x)\,dx$$

Answer: Use integration by parts with $u = \tan^{-1}(x)$ and $dv = dx$. The answer is

$$= x\tan^{-1}(x)\Big|_0^1 - \int_0^1 \frac{x}{x^2+1}\,dx = \frac{\pi}{4} - \frac{1}{2}\ln(2).$$

E) Use integration by parts to show this reduction formula: If n is a positive integer, then

$$\int x^n e^{ax}\, dx = \frac{x^n e^{ax}}{a} - \frac{n}{a}\int x^{n-1}e^{ax}\, dx$$

Solution: This comes directly from the integration by parts formula if you let $u = x^n$ and $dv = e^{ax}\, dx$, so $du = nx^{n-1}$ and $v = \frac{1}{a}e^{ax}$.

F) Apply the result from part (E) (repeatedly) to compute $\int x^4 e^{-2x}\, dx$.

Answer: $\left(\frac{-x^4}{2} - x^3 - \frac{3x^2}{2} - \frac{3x}{2} - \frac{3}{4}\right)e^{-2x} + C$.

II.

(A) Verify that

$$\int \csc\theta\, d\theta = \ln|\csc\theta - \cot\theta| + C$$

by differentiation.

Solution

$$\frac{d}{d\theta}\ln|\csc\theta - \cot\theta| = \frac{1}{\csc\theta - \cot\theta}\cdot(-\csc\theta\cot\theta + \csc^2\theta) = \csc\theta$$

so the given function is an antiderivative of $\csc\theta$.

(B) Which trigonometric substitution would you apply to compute $\int \frac{1}{u\sqrt{a^2-u^2}}\, du$? What trigonometric integral do you get after making the substitution? Complete the derivation of the integral.

Solution: Use $u = a\sin\theta$, so $du = a\cos\theta\, d\theta$. The integral goes over to

$$\int \frac{a\cos\theta\, d\theta}{a\sin\theta\cdot a\cos\theta} = \frac{1}{a}\int \csc\theta\, d\theta$$

By part A of this problem, this equals

$$\frac{1}{a}\ln|\csc(\theta) - \cot(\theta)| + C$$

Setting the triangle with u on the opposite side, a on the hypotenuse, the adjacent side is $\sqrt{a^2-u^2}$ and $\cot\theta = \frac{\sqrt{a^2-u^2}}{u}$. Hence

$$\int \frac{1}{u\sqrt{a^2-u^2}}\, du = \frac{1}{a}\ln\left|\frac{a}{u} - \frac{\sqrt{a^2-u^2}}{u}\right|.$$

(C) Our textbook's table of integrals gives this one as

$$\frac{-1}{a}\ln\left|\frac{a+\sqrt{a^2-u^2}}{u}\right|$$

Show that the form you got in part B is equivalent to this.

Solution: The form in the last line in part (B) can also be written as

$$\frac{1}{a} \ln \left| \frac{a - \sqrt{a^2 - u^2}}{u} \right|.$$

Multiply the top and the bottom of the fraction inside the logarithm by the conjugate radical $a + \sqrt{a^2 - u^2}$ and simplify.

III.

(A) Let R be the region in the plane bounded by $y = 3 - x^2$ and the x-axis.

(1) Sketch the region R.

Omitted – the region is extends from $x = -\sqrt{3}$ to $x = +\sqrt{3}$ under the parabola $y = 3 - x^2$, which opens down from the vertex at $(3/2, 3/4)$.

(2) Find the area of R.

Answer: $\int_{-\sqrt{3}}^{\sqrt{3}} 3 - x^2 \; dx = 4\sqrt{3}$.

(3) What is the average value of $f(x) = 3 - x^2$ on the interval where it takes positive values?

Answer: $\overline{y} = \frac{1}{2\sqrt{3}} \int_{-\sqrt{3}}^{\sqrt{3}} 3 - x^2 \; dx = 2$.

(4) Let R_1 be the region above the line $y = \overline{y}$ (the average value from (3)) and below $y = 3 - x^2$. Let R_2 be the region above the graph $y = 3 - x^2$ and below $y = \overline{y}$ over the interval where $3 - x^2 \geq 0$. How are the areas of R_1 and R_2 related. (Try to answer this without calculating!)

Answer: Those areas are *equal* because $2\sqrt{3} \cdot \overline{y} = 4\sqrt{3}$.

(5) Find the volume of the solid generated by rotating R about the x-axis.

Answer: $V = \int_{-\sqrt{3}}^{\sqrt{3}} \pi(3 - x^2)^2 \; dx = \frac{48\pi\sqrt{3}}{5}$.

(B) Let R be the region in the plane bounded by $y = 3x$ and $y = x^2$.

(1) Sketch the region R.

Omitted. The curves cross at $(0, 0)$ and $(3, 9)$.

(2) Find the area of R.

Answer: $\int_0^3 3x - x^2 \; dx = \frac{9}{2}$.

(3) Find the volume of the solid generated by rotating R about the x-axis.

Answer: The cross sections are washers with outer radius $3x$ and inner radius x^2 so $V = \int_0^3 \pi(3x)^2 - \pi(x^2)^2 \; dx = \frac{162\pi}{5}$.

(4) Find the volume of the solid generated by rotating R about the y-axis.

Answer: The cross-sections are washers again, but the outer radius at y is $x = \sqrt{y}$ and the inner radius is $x = y/3$. The volume is $V = \int_0^9 \pi(\sqrt{y})^2 - \pi(y/3)^2 \; dy = \frac{27\pi}{2}$.

(C) Let R be the region in the plane bounded by $y = \cos(\pi x)$, $y = 1/2$, $x = -1/3$ and $x = 1/3$.

(1) Sketch the region R.

Answer: Graph omitted. Since $\cos(\pi/3) = \cos(-\pi/3) = 1/2$, this is a lens-shaped area thickest at $x = 0$.

(2) Find the area of R.

Answer: $A = \int_{-1/3}^{1/3} \cos(\pi x) - \frac{1}{2}\, dx = \frac{\sqrt{3}}{\pi} - \frac{1}{3}$.

(3) Find the volume of the solid generated by rotating R about the x-axis.

(a) *Solution:* The cross-sections are washers with inner radius $1/2$ and outer radius $\cos(\pi x)$. The volume is

$$V = \int_{-1/3}^{1/3} \pi \cos^2(\pi x) - \pi(1/2)^2\, dx$$

Letting $u = \pi x$, we see we have to integrate $\int \cos^2 u\, du = \frac{u}{2} + \frac{1}{4}\sin(2u)$ by the double angle formula. So the volume is

$$V = \frac{\pi x}{2} + \frac{1}{4}\sin(2\pi x) - \frac{\pi x}{4}\Big|_{-1/3}^{1/3} = \frac{\sqrt{3}}{4} + \frac{\pi}{6}.$$

(4) Find the volume of the solid generated by rotating R about the line $y = -1$.

Answer:

$$V = \int_{-1/3}^{1/3} \pi(1 + \cos(\pi x))^2 - \pi(3/2)^2\, dx = \frac{9\sqrt{3}}{4} - \frac{\pi}{2}.$$

IV. The height of a monument is 20m. The horizontal cross-section of the monument at x meters from the top is an isosceles right triangle with legs $x/4$ meters. Is the given information enough to find the volume of the monument? If so, find the volume. If not, say why not.

Solution: By Cavalieri's Principle, this *is enough*. In particular, it doesn't actually matter how the cross-sections are stacked on top of each other. The volume is

$$V = \int_0^{20} \frac{1}{2}\left(\frac{x}{4}\right)^2\, dx = \frac{250}{3}.$$

V. A solid paperweight has a circular base of radius 4 cm. The cross-sections of the paperweight by planes perpendicular to one diameter of the base are equilateral triangles. Find the volume of the paperweight.

Solution: Let the given diameter be the segment of the x-axis from $x = -4$ to $x = 4$. The side of the slice at position x is then $2\sqrt{16 - x^2}$. The area of an equilateral triangle of side

s is $\frac{1}{2} \cdot s \cdot \frac{s\sqrt{3}}{2} = \frac{s^2\sqrt{3}}{4}$. By Cavalieri's Principle,

$$V = \int_{-4}^{4} \frac{4(16 - x^2)\sqrt{3}}{4}\, dx = \frac{256\sqrt{3}}{3}.$$

(The units would be cm^3.)

VI. A *45 rpm single* record was a vinyl disk 7 inches in diameter, with a large central hole 1 inch in diameter. A ring-shaped paper label 1 inch wide was usually glued to the vinyl surface outside the hole, but inside the grooves where the sound was recorded. The vinyl making up the disk, plus the paper of the label, had mass density .1 ounce per square inch at all points outside the hole in the central region. But the density then decreased linearly from .1 to .07 ounces per square inch at the outer edge. The density was the same at all points in the region the same distance away from the center of the hole. Set up an integral or integrals to compute the total mass of a 45 rpm single record (including the label) and compute the total mass.

Answer: The total mass is computed by

$$M = \int_{0}^{1/2} 0 \cdot 2\pi x\, dx + \int_{1/2}^{3/2} .1 \cdot 2\pi x\, dx + \int_{3/2}^{7/2} ((-.015)(x - 3/2) + .1) \cdot 2\pi x\, dx \doteq 3.24\text{oz}$$