Mathematics 136 – Calculus 2
Exam 1 – Review Sheet
September 20, 2016

Sample Exam Questions- Solutions

I.

(A) Since the interval is $[0,1]$ and $n = 4$, the sums are

$$L_4 = f(0)(.25) + f(.25)(.25) + f(.5)(.25) + f(.75)(.25) = .96875$$

$$R_4 = f(.25)(.25) + f(.5)(.25) + f(.75)(.25) + f(1)(.25) = 1.328125$$

$$M_4 = f(.125)(.25) + f(.375)(.25) + f(.625)(.25) + f(.875)(.25) = 1.71875$$

(B) $f'(x) = 2x + 2 > 0$ for all $x \in [0,1]$. Hence f is *increasing* on this interval. This implies that the left-hand Riemann sum is less than $\int_0^1 x^2 + 2x\,dx$, and the right-hand Riemann sum is greater than the value of the integral. Using the Evaluation Theorem, we can check this:

$$\int_0^1 x^2 + 2x\,dx = \frac{x^3}{3} + x^2\Big|_0^1 = \frac{4}{3} \doteq 1.3333$$

The left-hand sum is less and the right-hand sum is greater than this value.

(C) The sum is a right-hand Riemann sum for the function $f(x) = \frac{\cos(x)}{x}$ on the interval $[0,\pi]$. So (if the limit exists), it would compute the definite integral

$$\int_0^\pi \frac{\cos(x)}{x}\,dx.$$

(In point of fact, because $\frac{\cos(x)}{x}$ is not defined at $x = 0$, this is an "improper" integral and the limit does not exist in this case (it will not be a finite value – Oops!). We will return to this sort of question later in the semester.)

II.

A) The total distance traveled is approximated by

$$L_5 = \sqrt{5}\cdot 2 + \sqrt{7}\cdot 2 + \sqrt{9}\cdot 2 + \sqrt{11}\cdot 2 + \sqrt{13}\cdot 2 \doteq 29.61$$

B) The exact value is

$$\int_0^{10} \sqrt{t+5}\,dt = \frac{2}{3}(t+5)^{3/2}\Big|_0^{10} = \frac{2}{3}(15\sqrt{15} - 5\sqrt{5}) \doteq 31.28$$

III.

A) Compute $\lim_{N\to\infty}\sum_{i=1}^{N}\left(\frac{5i}{N}\right)^2\frac{5}{N}$ using the power sum formulas from page 265. (Note: If I ask a question like this, I would give those formulas – you don't need to memorize them.)

Solution: This sum equals:

$$\begin{aligned}
\sum_{i=1}^{N}\left(\frac{5i}{N}\right)^2\frac{5}{N} &= \frac{125}{N^3}\sum_{i=1}^{N} i^2 \\
&= \frac{125}{N^3}\left(\frac{N^3}{3}+\frac{N^2}{2}+\frac{N}{6}\right) \\
&= \frac{125}{3}+\frac{125}{2N}+\frac{125}{6N^2} \\
\Rightarrow \lim_{N\to\infty}\sum_{i=1}^{N}\left(\frac{5i}{N}\right)^2\frac{5}{N} &= \lim_{N\to\infty}\left(\frac{125}{3}+\frac{125}{2N}+\frac{125}{6N^2}\right) \\
&= \frac{125}{3}.
\end{aligned}$$

B)

$$\int_0^5 x^2\,dx = \left.\frac{x^3}{3}\right|_0^5 = \frac{125}{3}.$$

IV. Let

$$f(x) = \begin{cases} 1 & \text{if } 0 \le x \le 3 \\ x-2 & \text{if } 3 \le x \le 5 \\ 13-2x & \text{if } 5 \le x \le 8 \end{cases}$$

(A) Sketch the graph $y = f(x)$. The graph is made up of segments of three different straight lines. See figure at top of next page.

In the rest of the parts, $F(x) = \int_0^x f(t)\,dt$, where f is the function from part A.

(B) Assuming $F(0) = 0$, Compute $F(1), F(2), F(3), F(4), F(5), F(6), F(7), F(8)$ given the information in the graph of f.

Using the area interpretation of the definite integral we have

$$F(1) = \int_0^1 f(x)\,dx = 1$$

$$F(2) = \int_0^2 f(x)\,dx = 2$$

$$F(3) = \int_0^3 f(x)\,dx = 3$$

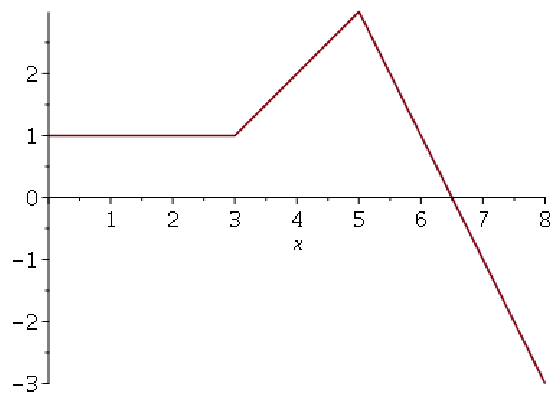

Figure 1: Figure for Problem IV.

$$F(4) = \int_0^3 f(x)\,dx + \int_3^4 f(x)\,dx = 3 + \frac{3}{2} = \frac{9}{2}$$

$$F(5) = \int_0^4 f(x)\,dx + \int_4^5 f(x)\,dx = \frac{9}{2} + \frac{5}{2} = 7$$

$$F(6) = \int_0^5 f(x)\,dx + \int_5^6 f(x)\,dx = 7 + 2 = 9$$

$$F(7) = \int_0^6 f(x)\,dx + \int_6^{13/2} f(x)\,dx + \int_{13/2}^7 f(x)\;dx = 9 + \frac{1}{4} - \frac{1}{4} = 9$$

$F(8) = \int_0^5 f(x)\,dx + \int_5^{13/2} f(x)\,dx + \int_{13/2}^8 f(x)\,dx = \int_0^5 f(x)\,dx = 7$ (the last two integrals cancel since they represent equal areas with opposite signs).

(C) By the Fundamental Theorem of Calculus, $F'(x) = f(x)$. Since $f(13/2) = 0$, the point $x = 13/2$ is a critical point. Since $F' = f$ changes sign from positive to negative at the critical point, $x = 13/2$ is a local maximum.

(D) Since $G(x) = \int_2^x f(t)\;dt = \int_0^x f(t)\;dt - \int_0^2 f(t)\;dt = F(x) - 2$, the graph $y = G(x)$ is obtained from $y = F(x)$ by shifting down 2 units along the y-axis.

V. Find the derivatives of the following functions

(A) $f(x) = \int_0^x \sin(t)/t\,dt$.

$$f'(x) = \frac{\sin x}{x}$$

(B) $g(x) = \int_5^{x^3} \tan^4(t)\ dt$.

$g(x) = m(x^3)$, where $m(x) = \displaystyle\int_5^x \tan^4(t)\ dt$. Then, $g'(x) = m'(x^3) \cdot 3x^2 = \tan^4(x^3) \cdot 3x^2$.

(C)) $h(x) = \int_{-3x}^{5x} e^{t^2} \sin(t)\ dt$.

$h(x) = n(x) + l(x)$, where $n(x) = \displaystyle\int_{-3x}^{0} e^{t^2} \sin(t)\ dt$ and $l(x) = \displaystyle\int_0^{5x} e^{t^2} \sin(t)\ dt$. Then $h'(x) = n'(x) + l'(x) = -(e^{(-3x)^2} \sin(-3x)) \cdot (-3) + 5 \cdot e^{(5x)^2} \sin(5x) = 3e^{9x^2} \sin(-3x) + 5e^{25x^2} \sin(5x)$.

VI.

(A) Compute $\int 5x^4 - 3\sqrt{x} + e^x + \frac{2}{x}\ dx$

$$\int 5x^4 - 3\sqrt{x} + e^x + \frac{2}{x}\ dx = x^5 - 2x^{3/2} + e^x + 2\ln|x| + C$$

(B) Apply a u-substitution to compute $\int x(4x^2-3)^{3/5}\ dx$

$u = 4x^2 - 3,\ du = 8x\,dx$. Then $\displaystyle\int x(4x^2-3)^{3/5}\ dx = \int \frac{1}{8} u^{3/5}\,dx = \frac{1}{8}\frac{u^{8/5}}{8/5} + C = \frac{5}{64}(4x^2-3)^{8/5} + C$

(C) Apply a u-substitution to compute $\int_1^2 e^{\sin(\pi x)} \cos(\pi x)\ dx$

$u = \sin(\pi x),\ du = \pi\cos(\pi x)$. Then $\displaystyle\int_1^2 e^{\sin(\pi x)} \cos(\pi x)\ dx = \frac{1}{\pi}\int_0^0 e^u\,du = 0$

(D) Let $u = t^3 + 3t + 3$, then $du = (3t^2+3)\ dt = 3(t^2+1)\ dt$. Then given integral is

$$\int \frac{t^2+1}{t^3+3t+3}\ dt = \int \frac{1}{3u}\ du = \frac{1}{3}\ln|u| + C = \frac{1}{3}\ln|t^3+3t+3| + C.$$

(E) Let $u = -7x$ or "guess and check." *Answer*: $\frac{-1}{7}e^{-7x} + C$.

VII. Compute each of the integrals below.

(A) Let $u = \sqrt{\sin(x)}$. Then $du = \frac{\cos(x)}{2\sqrt{\sin(x)}}\ dx$ by the Chain Rule. So the given integral is

$$2\int e^u\ du = 2e^u + C = 2e^{\sqrt{\sin(x)}} + C.$$

(B) One solution: Let $u = \sin(2x)$. Then $du = 2\cos(2x)\ dx$, so the given integral is

$$\int u \cdot \frac{1}{2}\ du = \frac{1}{4}u^2 + C = \frac{1}{4}\sin^2(2x) + C.$$

You can equally well let $u = \cos(2x)$ and then $du = -2\sin(2x)\ dx$ and the given integral is

$$\int u \cdot \frac{-1}{2}\ du = -\frac{1}{4}u^2 + C = -\frac{1}{4}\cos^2(2x) + C.$$

Both answers are correct, and each differs from the other by an additive constant because of the identity $\cos^2(2x) + \sin^2(2x) = 1$ for all x.