MATH 135 – Calculus 1
Derivatives of Inverse Functions
November 7, 2016

Background

The method of implicit differentiation is important because it gives us one way to compute derivatives of inverse functions of functions we know. We'll illustrate this first with the derivative formula for $\ln(x)$, which is the inverse function of e^x. The definition of an inverse function shows that if $y = \ln(x)$, then $e^y = e^{\ln(x)} = x$. We can then compute $\frac{dy}{dx}$ by implicit differentiation from the equation: $e^y = x$. This gives

$$e^y \frac{dy}{dx} = 1 \Rightarrow \frac{dy}{dx} = \frac{1}{e^y} = \frac{1}{x}.$$

Hence,

$$\frac{d}{dx}\ln(x) = \frac{1}{x} \qquad \text{and} \qquad \frac{d}{dx}\ln(u) = \frac{1}{u}\frac{du}{dx}, \tag{1}$$

if u is a function of x, by the Chain Rule.

Similarly, if $y = \sin^{-1}(x)$ (the inverse function of sin with domain $[-1, 1]$ and range $[-\pi/2, \pi/2]$), we have $\sin(y) = x$, so by implicit differentiation,

$$\cos(y)\frac{dy}{dx} = 1 \Rightarrow \frac{dy}{dx} = \frac{1}{\cos(y)} = \frac{1}{\sqrt{1-x^2}}$$

(with some trigonometric identities). Hence,

$$\frac{d}{dx}\sin^{-1}(x) = \frac{1}{\sqrt{1-x^2}} \qquad \text{and} \qquad \frac{d}{dx}\sin^{-1}(u) = \frac{1}{\sqrt{1-u^2}}\frac{du}{dx}, \tag{2}$$

if u is a function of x, by the Chain Rule.

Finally, if $y = \tan^{-1}(x)$ (the inverse function of tan with domain $(-\infty, \infty)$ and range $(-\pi/2, \pi/2)$), we have $\tan(y) = x$, so by implicit differentiation,

$$\sec^2(y)\frac{dy}{dx} = 1 \Rightarrow \frac{dy}{dx} = \frac{1}{\sec^2(y)} = \frac{1}{1+x^2},$$

again using some trig identities. Hence,

$$\frac{d}{dx}\tan^{-1}(x) = \frac{1}{1+x^2} \qquad \text{and} \qquad \frac{d}{dx}\tan^{-1}(u) = \frac{1}{1+u^2}\frac{du}{dx}, \tag{3}$$

if u is a function of x, by the Chain Rule.

Questions

Compute derivative of the following $f(x)$ using (1), (2), (3) and our other derivative rules:

1. $f(x) = \ln(x^2 + x + 1)$
2. $f(x) = x\sin^{-1}(x^2)$ (Use (2), the product rule and the chain rule.)
3. $f(x) = \dfrac{x^2}{\tan^{-1}(x)}$ (Use (3) and the quotient rule.)
4. $f(x) = \tan^{-1}(e^x)$.