MATH 135 – Calculus 1
The Intermediate Value Theorem
October 7, 2016

Background

It's our lucky day, because today we get to discuss some beautiful *real mathematics*(!) The Intermediate Value Theorem (IVT) is the following statement:

Theorem 1 (IVT) *Let $f(x)$ be a function that is continuous at every x in a closed interval $[a, b]$. Then for every M between and $f(a)$ and $f(b)$, there exists at least one $x = c$ in the interval where $f(c) = M$.*

What this is saying in intuitive terms is that the graph of a function that is continuous at every point of a closed interval *really is one unbroken curve* that can be drawn without lifting your pen or pencil from the paper. In the process it goes through every y-value between the y-coordinates of the endpoints $(a, f(a))$ and $(b, f(b))$. This statement has some important and surprising consequences!

Questions

(1) An important consequence: The IVT implies that equations have solutions(!) The IVT implies, for instance, that every cubic polynomial

$$P(x) = Ax^3 + Bx^2 + Cx + D$$

(where $A \neq 0$) has some real root. Here's why:

(a) Since $A \neq 0$ we can divide both sides of the equation $Ax^3 + Bx^2 + Cx + D = 0$ by A to get an equation of the form $x^3 + \beta x^2 + \gamma x + \delta = 0$, where $\beta = B/A$, etc. Explain why, no matter what β, γ, δ are, you can always find a really big $x = b > 0$ where $b^3 + \beta b^2 + \gamma b + \delta > 0$. On the other hand there must also be really negative $x = a < 0$ where $a^3 + \beta a^2 + \gamma a + \delta < 0$.

(b) Then think about what the IVT says about the cubic polynomial $x^3 + \beta x^2 + \gamma x + \delta$ on the interval $[a, b]$ between the very negative number a and the very positive number b. (Hint: We want to say $c^3 + \beta c^2 + \gamma c + \delta = 0$ for some $x = c$ in the interval. Why does that follow?)

(2) A surprising consequence. Take any map and draw a circle on it anywhere. (See Figure 6 on page 103 of our textbook, for example.) I claim that the there are two distinct points on that circle where the temperature at the current instant is *exactly the same*. Here's the idea: If we put in a coordinate system with the center of the circle at $(0, 0)$, then we can measure the counterclockwise angle θ from the positive x-axis to any diameter of the circle. Let $f(\theta)$ be *the difference between the temperatures at the two endpoints of the diameter*. In other words, if A and B are the two endpoints as in the figure, then $f(\theta)$ = temperature at A minus the temperature at B. It is reasonable to assume that $f(\theta)$ is a continuous function of θ (Why?). Then consider what happens as θ increases from 0 to π radians. How are the values $f(0)$ and $f(\pi)$ related? What does the IVT tell you about f on the interval $[0, \pi]$?