MATH 135 – Calculus 1
Solutions/Answers for Exam 3 Practice Problems
November 18, 2016

I. Find the indicated derivative(s) and simplify.

(A)

$$y = \ln(x)\left(x^7 - \frac{4}{\sqrt{x}}\right)$$

Solution: By the product rule and the derivative rules for $\ln(x)$ and powers:

$$y' = \ln(x)(7x^6 + 2x^{-3/2}) + x^6 - \frac{4}{x^{3/2}}$$

(B)

$$y = \sin^{-1}(e^{2x} + 2)$$

Solution: By the chain rule and the inverse sine derivative rule:

$$y' = \frac{1}{\sqrt{1 - (e^{2x}+2)^2}} \cdot 2e^{2x} = \frac{2e^{2x}}{\sqrt{1 - (e^{2x}+2)^2}}$$

(C)

$$y = \frac{\ln(x+1)}{3x^4 - 1}$$

Solution: Using the quotient rule:

$$y' = \frac{(3x^4 - 1) \cdot \frac{1}{x+1} - \ln(x+1)(12x^3)}{(3x^4-1)^2}$$

(D)

$$y = \frac{\sin(x)}{1 + \cos(x)}$$

Solution: By the quotient rule:

$$y' = \frac{(1 + \cos(x))\cos(x) + \sin^2(x)}{(1+\cos(x))^2} = \frac{1}{1 + \cos(x)}$$

(E)

$$y = \tan^{-1}(x^2 + x)$$

Solution: By the inverse tangent rule:

$$\frac{dy}{dx} = \frac{2x+1}{1 + (x^2 + 2x)^2}$$

(F) Using implicit differentiation:

$$xy^2 - 3y^3 + 2x^4 - 4xy = 2$$

Solution: We have

$$2xy\frac{dy}{dx} + y^2 - 9y^2\frac{dy}{dx} + 8x^3 - 4x\frac{dy}{dx} - 4y = 0$$

so

$$\frac{dy}{dx} = \frac{4y - 8x^3 - y^2}{2xy - 9y^2 - 4x}$$

(G) Find the equation of the line tangent to the curve from (F) at $(x, y) = (1, 0)$

Solution: The slope is 2, so $y = 2x - 2$.

(H) Find

$$\frac{d}{dx}\left(5x\sqrt{x} - \frac{2}{x^3} + 11x - 4\right)$$

Solution: The function can also be written as $5x^{3/2} - 2x^{-3} + 11x - 3$. In this form, we only need the power rule to differentiate:

$$y' = \frac{15}{2}x^{1/2} + 6x^{-4} + 11.$$

(I)

$$\frac{d}{dt}\left(\frac{t^2e^{3t}}{t^4 + 1}\right)$$

Solution: By the quotient rule, product rule, and chain rule the derivative is:

$$\frac{(t^4 + 1) \cdot (3t^2e^{3t} + 2te^{3t}) - (t^2e^{3t}) \cdot (4t^3)}{(t^4 + 1)^2} = \frac{e^{3t}(3t^6 - 2t^5 + 3t^2 + 2t)}{(t^4 + 1)^2}.$$

(J)

$$\frac{d^2}{dz^2}\frac{z^2 - 2z + 4}{z^2 + 1}$$

Solution: Again using the quotient rule, the first derivative is:

$$\frac{(z^2 + 1)(2z - 2) - (z^2 - 2z + 4)(2z)}{(z^2 + 1)^2} = \frac{2z^2 - 6z - 2}{(z^2 + 1)^2}.$$

So then differentiating again, the second derivative is

$$\begin{aligned}
&= \frac{(z^2 + 1)^2(4z - 6) - (2z^2 - 6z - 2)(2)(z^2 + 1)(2z)}{(z^2 + 1)^4} \\
&= \frac{(z^2 + 1)(4z - 6) - (4z)(2z^2 - 6z - 2)}{(z^2 + 1)^3} \\
&= \frac{-4z^3 + 18z^2 + 12z - 6}{(z^2 + 1)^3}.
\end{aligned}$$

(K)

$$\frac{d}{dx}\left(\sin(x)\left(x^7 - \frac{4}{\sqrt{x}}\right)\right)$$

Solution: Rewrite the fuction as $\sin(x)(x^7 - 4x^{-1/2})$. Then by the product rule the derivative is:

$$\sin(x)(7x^6 + 2x^{-3/2}) + (x^7 - 4x^{-1/2})\cos(x).$$

(L) Find y' (note this is just another way of asking the same question!)

$$y = (e^{2x} + 2)^3$$

By the chain rule, the derivative is:

$$3(e^{2x} + 2)^2(2e^{2x}) = 6e^{2x}(e^{2x} + 2)^3.$$

(M) Find y' and y''

$$y = \frac{x + 1}{3x^4 - 1}$$

Solution: By the quotient rule,

$$y' = \frac{(3x^4 - 1)(1) - (x + 1)(12x^3)}{(3x^4 - 1)^2} = \frac{-9x^4 - 12x^3 - 1}{(3x^4 - 1)^2}.$$

So then differentiating again with the quotient rule, we get

$$y'' = \frac{(12x^2)(9x^5 + 15x^4 + 5x + 3)}{(3x^4 - 1)^3}$$

(There is a common factor of $3x^4 - 1$ that can be cancelled between the numerator and the denominator after you apply the quotient rule the second time.)

(N) Find y'

$$y = \frac{\sin(x)}{1 + \cos(x)} + x^2\cos(x^3 + 3)$$

Solution: By the quotient, product, and chain rules:

$$\begin{aligned} y' &= \frac{(1 + \cos(x))\cos(x) - \sin(x)(-\sin(x))}{(1 + \cos(x))^2} - x^2\sin(x^3 + 3)(3x^2) + 2x\cos(x^3 + 3) \\ &= \frac{1 + \cos(x)}{(1 + \cos(x))^2} - 3x^4\sin(x^3 + 3) + 2x\cos(x^3 + 3) \\ &= \frac{1}{1 + \cos(x)} - 3x^4\sin(x^3 + 3) + 2x\cos(x^3 + 3). \end{aligned}$$

II. The total cost (in \$) of repaying a car loan at interest rate of r% per year is $C = f(r)$.

(A) What is the meaning of the statement $f(7) = 20000$?

Solution: At an interest rate of 7% per year, the cost of repaying the loan is 20000 dollars.

(B) What is the meaning of the statement $f'(7) = 3000$? What are the units of $f'(7)$?

Solution: At an interest rate of 7% per year, the rate of change of the cost of repaying the loan is 3000 dollars per (% per year).

III. The quantity of a reagent present in a chemical reaction is given by $Q(t) = t^3 - 3t^2 + t + 30$ grams at time t seconds for all $t \geq 0$. (Note: For a question like this, I could also give you the plot of the function and ask questions like those below. In this case you need to start from the formula and compute $Q'(t)$; if you were given the graph, you need to make the connection between slopes of tangent lines and signs of $Q'(t)$ visually.)

(A) Over which intervals with $t \geq 0$ is the amount increasing? (i.e. $Q'(t) > 0$) decreasing (i.e. $Q'(t) < 0$)?

Solution: $Q'(t) = 3t^2 - 6t + 1$. $Q'(t) = 0$ when

$$t = \frac{6 \pm \sqrt{36 - 12}}{6} = 1 \pm \frac{\sqrt{6}}{3} \doteq 1.816, .184.$$

Since this is a quadratic function with a positive t^2 coefficient, $Q'(t) > 0$ for $t > 1.816$ and $t < .184$. $Q'(t) < 0$ for $.184 < t < 1.816$ (t in seconds).

(B) Over which intervals is the rate of change of Q increasing? decreasing?

Solution: The rate of change of Q is increasing when $(Q')' > 0$ and decreasing when $(Q')' < 0$. The second derivative of Q is $Q''(t) = 6t - 6$. So $Q''(t) > 0$ for $t > 1$ and $Q''(t) < 0$ for $t < 1$ (t in seconds).

IV. A spherical balloon is being inflated at 20 cubic inches per minute. When the radius is 6 inches, at what rate is the radius of the balloon increasing? At what rate is the surface area increasing? (The volume of a sphere of radius r is $V = \frac{4\pi r^3}{3}$ and the surface area is $A = 4\pi r^2$.)

Solution: By the chain rule, $\frac{dV}{dt} = 4\pi r^2 drdt$. We are given $\frac{dV}{dt} = 20$ when $r = 6$, so

$$\frac{dr}{dt} = \frac{20}{4\pi(6)^2} = \frac{5}{36\pi}$$

inches per minute. The rate of change of the surface area is

$$\frac{dA}{dt} = 8\pi r\frac{dr}{dt} = \frac{48\pi \cdot 5}{36\pi} = \frac{20}{3}$$

square inches per minute.

V. A baseball diamond is a square with side of length 90 feet. After hitting the ball, a player leaves home plate and runs toward first base at 15 ft/sec. (Assume the runner is running straight along the base path – this is a bit unrealistic, of course, but let's keep it simple for the purposes of the problem!) How fast is the runner's distance from second base changing when he is half way to first base?

Solution: Let x be the distance traveled by the runner along the basepath. From a diagram of the diamond, we can see that the runner's position, first base, and second base form a right triangle (with right angle at first base) at all times up until the runner reaches first base. The two legs of that triangle are 90 and $90 - x$, so by the Pythagorean theorem, the distance from the runner to second base is

$$\ell = \sqrt{90^2 + (90 - x)^2} = \sqrt{16200 - 180x + x^2}.$$

Take derivatives with respect to t (time) everywhere. Then

$$\frac{d\ell}{dt} = \frac{1}{2}(16200 - 180x + x^2)^{-1/2}(2x - 180)\frac{dx}{dt}$$

The given value 15 ft/sec is the $\frac{dx}{dt}$, and we want the instant when $x = 45$. So at that time,

$$\frac{d\ell}{dt} = \frac{1}{2}(16200 - 180 \cdot 45 + 45^2)^{-1/2}(-90)(15) = -\frac{15}{\sqrt{5}}$$

ft/sec. (This is negative, so the distance from the runner to second base is *decreasing.*)

VI. All parts of this question refer to $f(x) = 4x^3 - x^4$.

(A) Find and classify all the critical points of f using the First Derivative Test.

Solution: $f'(x) = 12x^2 - 4x^3 = 4x^2(3 - x)$. This is defined for all x and equal to zero at $x = 0$ and $x = 3$. Note that $4x^2 \geq 0$ for all x. So the sign of $f'(x)$ comes from the $3 - x$ factor. That is negative for $x > 3$ and positive for $x < 3$. Hence f' changes sign from positive to negative at $x = 3$ and the First Derivative Test says f has a local local maximum at $x = 3$. On the other hand, $f'(x)$ does not change sign at $x = 0$, so that critical point is neither a local maximum nor a local minimum.

(B) Over which intervals is the graph $y = f(x)$ concave up? concave down?

Solution: $f''(x) = 24x - 12x^2 = 12x(2 - x)$, which is zero at $x = 0$ and $x = 2$. Then $f''(x) > 0$ and the graph $y = f(x)$ is concave up on $(0, 2)$ and $f''(x) < 0$ and the graph $y = f(x)$ is concave down on $(-\infty, 0)$ and $(2, \infty)$.

(C) Sketch the graph $y = f(x)$.

Solution: See Figure 1 on the back of this page.

(D) Find the absolute maximum and minimum of $f(x)$ on the interval $[1, 4]$.

Solution: Only the critical point $x = 3$ is in this interval. $f(1) = 8$, $f(3) = 27$ and $f(4) = 0$. So $f(3) = 27$ is the maximum value and $f(4) = 0$ is the minimum value on the interval $[1, 4]$.

VII. All three parts of this question refer to the function $f(x)$ *whose derivative is plotted in Figure 1. NOTE: This is the graph* $y = f'(x)$ *not* $y = f(x)$.

(A) Give approximate values for all the critical points of $f(x)$ in the interval shown, and say whether f has a local maximum, a local minimum, or neither at each.

Solution: By inspection of the plot in Figure 1, we see that $f'(x) = 0$ at approximately $x = -5.2, -2$, and 1.2. Since f' changes sign from $+$ to $-$ at $x = -5.2$ and $x = 1.2$, those are local maxima for f. Since f' changes sign from negative to positive at $x = -2$, that is a local minimum for f.

(B) Find approximate values for all the inflection points of $f(x)$.

Solution: $y = f(x)$ has inflection points where f' changes from increasing to decreasing. That happens here at roughly $x = -3.9$ and $x = -0.8$.

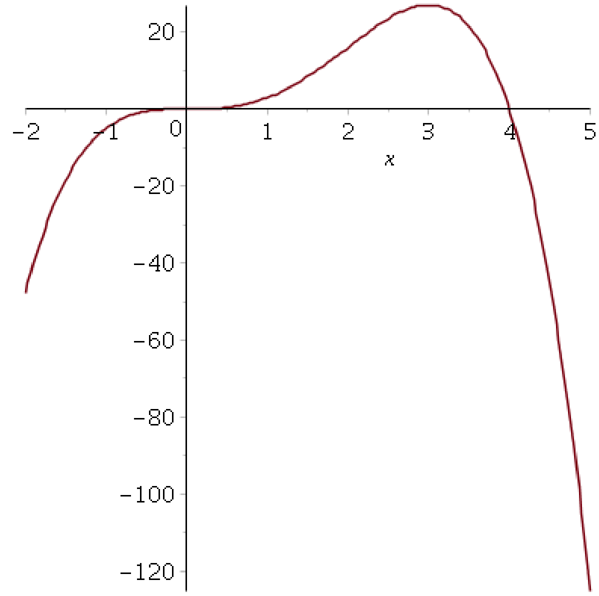

Figure 1: Plot of $y = f(x)$ for Problem VI

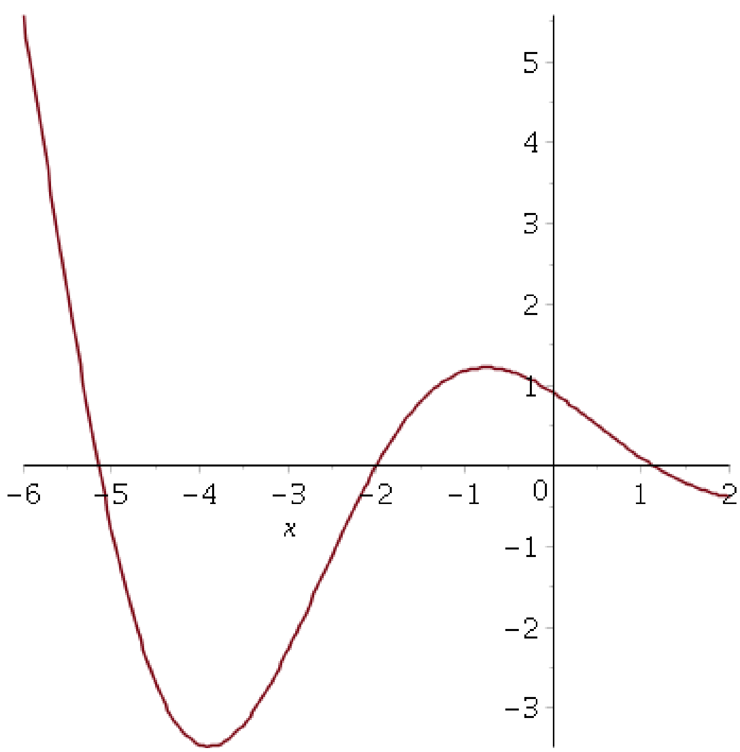

Figure 2: Plot of $y = f'(x)$ for Problem VII

(C) Over which intervals is $y = f(x)$ concave up? concave down?

Solution: Following on from (B), $y = f(x)$ will be concave down on intervals where $f'(x)$ is decreasing – roughly $(-6, -3.9)$ and $(-0.8, 2)$. $y = f(x)$ will be concave up on intervals where $f'(x)$ is increasing – roughly $(-3.9, -0.8)$.