Mathematics 136 – Calculus 2
Summary of Trigonometric Substitutions
Spring 2014

The trigonometric substitution method handles many integrals containing expressions like

$$\sqrt{a^2 - x^2}, \sqrt{x^2 + a^2}, \sqrt{x^2 - a^2}$$

(possibly including expressions without the square roots!) The basis for this approach is the trigonometric identities

$$1 = \sin^2\theta + \cos^2\theta$$
$$\Rightarrow \sec^2\theta = \tan^2\theta + 1.$$

from which we derive other related identities:

$$\sqrt{a^2 - (a\sin\theta)^2} = \sqrt{a^2(1 - \sin^2\theta)} = \sqrt{a^2\cos^2\theta} = a\cos\theta$$
$$\sqrt{(a\tan\theta)^2 + a^2} = \sqrt{a^2(\tan^2\theta + 1)} = \sqrt{a^2\sec^2\theta} = a\sec\theta$$
$$\sqrt{(a\sec\theta)^2 - a^2} = \sqrt{a^2(\sec^2\theta - 1)} = \sqrt{a^2\tan^2\theta} = a\tan\theta$$

(Technical note: We usually assume that $a > 0$ and $0 < \theta < \pi/2$ here so that all the trig functions take positive values.) Hence,

1. If our integral contains $\sqrt{a^2 - x^2}$, the substitution $x = a\sin\theta$ will convert this radical to the simpler form $a\cos\theta$.
2. If our integral contains $\sqrt{x^2 + a^2}$, the substitution $x = a\tan\theta$ will convert this radical to the simpler form $a\sec\theta$.
3. If our integral contains $\sqrt{x^2 - a^2}$, the substitution $x = a\sec\theta$ will convert this radical to the simpler form $a\tan\theta$.

We substitute for the rest of the integral *including the* dx. For instance if $x = a\sin\theta$, then the $dx = a\cos\theta\ d\theta$. If $x = a\tan\theta$, then $dx = a\sec^2\theta\ d\theta$. If $x = a\sec\theta$, then $dx = a\sec\theta\tan\theta\ d\theta$. All these substitutions should produce an integral with no x terms left – everything expressed in terms of θ. If desired, limits of integration on definite integrals can also be converted to equivalent θ-values.

We then integrate the resulting trigonometric form using the trig methods discussed last time, and convert back to the original variable (or substitute limits of integration in terms of θ).

Two Examples

A) Compute $\int \frac{u^2}{\sqrt{a^2-u^2}}\ du$. Solution: The $\sqrt{a^2 - u^2}$ tells us that we want the sine substitution: $u = a\sin\theta$. Then $du = a\cos\theta\ d\theta$, and the integral becomes:

$$\int \frac{a^3\sin^2\theta\cos\theta\ d\theta}{a\cos\theta} = a^2\int \sin^2\theta\ d\theta$$

We apply the half-angle formula $\sin^2\theta = \frac{1}{2}(1-\cos(2\theta))$ to this and integrate

$$a^2 \int \frac{1}{2}(1-\cos(2\theta))\,d\theta$$

to yield

$$= \frac{a^2}{2}\theta - \frac{a^2}{4}\sin(2\theta) = \frac{a^2\theta}{2} - \frac{a^2}{2}\sin\theta\cos\theta + C.$$

Then, we convert back to functions of u using the substitution equation $u = a\sin\theta$. From this,

$$\theta = \arcsin(u/a), \qquad \cos\theta = \sqrt{a^2-u^2}/a, \qquad \sin\theta = u/a$$

so the integral is

$$\int \frac{u^2}{\sqrt{a^2-u^2}}\,du = \frac{a^2}{2}\arcsin(u/a) - \frac{1}{2}u\sqrt{a^2-u^2} + +C.$$

B) Compute $\int \frac{dx}{\sqrt{x^2+16}}$. Solution: The $\sqrt{x^2+16}$ indicates that we want the tangent substitution $x = 4\tan\theta$. Then $dx = 4\sec^2\theta\ d\theta$ and the integral becomes:

$$\int \frac{4\sec^2\theta\ d\theta}{4\sec\theta} = \int \sec\theta\ d\theta.$$

As we saw last time, this form can be integrated as follows:

$$\ln|\sec\theta + \tan\theta| + C$$

Then, from $x = 4\tan\theta$, we get $\sec\theta = \frac{\sqrt{x^2+16}}{4}$ and $\tan\theta = \frac{x}{4}$. Hence the integral equals:

$$\ln\left|\frac{\sqrt{x^2+16}}{x} + \frac{4}{x}\right| + C.$$

Note: If this was a definite integral, say

$$\int_0^4 \frac{dx}{\sqrt{x^2+16}},$$

then the equivalent limits of integration in terms of θ would be found from the formula $\theta = \tan^{-1}(x/4)$, so $x = 0 \Leftrightarrow \theta = 0$ and $x = 4 \Leftrightarrow \theta = \pi/4$. If we used these values, then we could apply the Evaluation Theorem to the trig integral, and it would not be necessary to convert back to the antiderivative in terms of x.