MATH 136, 1 PS 9 'B' Solutions

§7.5/18. the Gompertz equation $\frac{dP}{dt} = c\ln\left(\frac{M}{P}\right)P$

(a) this is separable:

$$\int \frac{dP}{\ln\left(\frac{M}{P}\right)P} = \int c\,dt \qquad \text{let } u = \ln\left(\frac{M}{P}\right)$$

$$du = \frac{P}{M}\cdot\frac{-M\,dP}{P^2} = -\frac{1}{P}dP$$

$$\Rightarrow -\int\frac{du}{u} = \int c\,dt$$

so $-\ln\left|\ln\left(\frac{M}{P}\right)\right| = ct + k$ some const. k

$\therefore \ln\left(\frac{M}{P}\right) = \ell e^{-ct}$ $\quad (\ell = \pm e^{-k})$

$\therefore$ $\boxed{P(t) = M\exp(-\ell e^{-ct})}$ $\quad (\exp(x) = e^x)$

If $P(0) = P_0$ is known, $P_0 = M\exp(-\ell)$

so $\ell = \ln\left(\frac{M}{P_0}\right)$, and

$$\boxed{P(t) = M\exp\left(\ln\left(\frac{P_0}{M}\right)e^{-ct}\right)}$$

(b) Assuming $c > 0$, $\lim_{t\to\infty} e^{-ct} = 0$, so

$$\lim_{t\to\infty} P(t) = M.$$

(c) From Example 3, $M = 1000$, $P(0) = 100$

$P_{log}(t) = \frac{1000}{1+9e^{-.08t}}$ is the solution of the logistic equation. Here $c = .05$ is given

$$P_{Gom}(t) = 1000\exp\left(\ln(.1)\,e^{-.05t}\right)$$

these are plotted together on the next page

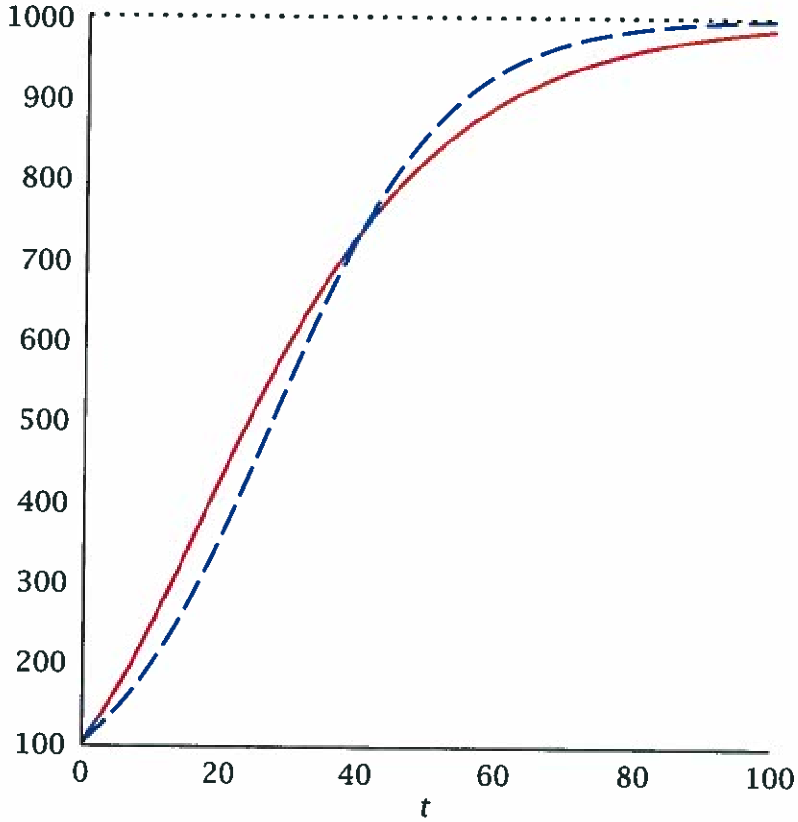
$Gom := t \to 1000 \cdot \exp(\ln(.1) \cdot \exp(-.05 \cdot t));$

$$t \to 1000\, e^{\ln(0.1)\, e^{(-1) \cdot 0.05\, t}} \tag{1}$$

$Logistic := t \to \dfrac{1000}{1 + 9 \cdot \exp(-.08 \cdot t)};$

$$t \to \frac{1000}{1 + 9\, e^{(-1) \cdot 0.08\, t}} \tag{2}$$

plot([*Gom*(*t*), *Logistic*(*t*), 1000], *t* = 0..100, *color* = [*red*, *blue*, *black*], *linestyle* = [*solid*, *dash*, *dot*]);



The dashed blue curve is the solution of the logistic equation; the solid red curve is the solution of the Gompertz equation. Note that the Gopertz solution has a similar overall shape, but it it increases faster at first, and then approaches the horizontal asymptote at P = 1000 more slowly.

(d) $f(P) = c \ln\left(\frac{M}{P}\right) P$ is largest when

$$0 = f'(P) = c \cdot \left[\frac{P}{M} \cdot \frac{-M}{P^2} \cdot P - \ln\left(\frac{m}{P}\right)\right]$$

$$= c\left(1 - \ln\left(\frac{m}{P}\right)\right).$$

this is true when $\ln\left(\frac{m}{P}\right) = 1$, so $\frac{m}{P} = e$,

or $P = \frac{M}{e}$.

§ 7.4/21

(a) A separable equation:

$$\int \frac{dP}{P - \frac{m}{k}} = \int k\, dt$$

$$\ln\left|P - \frac{m}{k}\right| = kt + c$$

so $P = \frac{m}{k} + \ell e^{kt}$. $\quad (\ell = \pm e^{c})$

If $P(0) = P_0$ is known,

$P_0 = \frac{m}{k} + \ell$, so $\ell = P_0 - \frac{m}{k}$

$$\boxed{P(t) = \frac{m}{k} + \left(P_0 - \frac{m}{k}\right) e^{kt}}$$

(b) In order to get exponential growth, $k = \alpha - \beta > 0$

and $\boxed{kP_0 > m}$, or $P_0 - \frac{m}{k} > 0$

(c) the population is constant if $P_0 = \frac{m}{k}$, or

$\boxed{m = kP_0}$; the population declines if $\boxed{kP_0 < m}$,

or $P_0 - \frac{m}{k} < 0$

(d) With $P_0 = 8{,}000{,}000$, $k = .016$, $m = 210{,}000$,

we have $\boxed{m > kP_0}$, so the population will decline

§8.2 / 58 We see $\angle CAD = \angle ECD = \angle EDF = \cdots$, so

$$|CD| + |DE| + |EF| + |FG| + \cdots$$

$$= b\sin\theta + b\sin^2\theta + b\sin^3\theta + b\sin^4\theta + \cdots$$

this is a geometric series with $\boxed{a = b\sin\theta}$, $\boxed{r = \sin\theta}$.

Since $0 < \theta < \frac{\pi}{2}$, $|\sin\theta| < 1$, and the series

converges to $\boxed{\dfrac{b\sin\theta}{1-\sin\theta}}$.

§8.3 / 32

(a) $S_{10} = \frac{1}{1^4} + \frac{1}{2^4} + \cdots + \frac{1}{10^4} \doteq 1.082036583$

From (3), p. 581, $\int_{11}^{\infty} \frac{1}{x^4} \leq \text{error} \leq \int_{10}^{\infty} \frac{1}{x^4}$, so $.0002504 \leq \text{error} \leq .000\overline{3}$

(b) From equation (4), p. 582, we get

$$S_{10} + \int_{11}^{\infty} \frac{1}{x^4}\,dx \leq S \leq S_{10} + \int_{10}^{\infty} \frac{1}{x^4}\,dx$$

we have $\displaystyle\int_{11}^{\infty} \frac{1}{x^4}\,dx = \frac{-1}{3}\frac{1}{x^3}\Big|_{11}^{\infty} = \lim_{b\to\infty} \frac{1}{3\cdot(11)^3} - \frac{1}{3b^3}$

$$\doteq .0002504$$

$$\int_{10}^{\infty} \frac{1}{x^4}\,dx = \lim_{b\to\infty} \frac{1}{3(10)^3} - \frac{1}{3b^3}$$

$$\doteq .000\overline{3}$$

So $1.08203 + .00025 \leq S \leq 1.08203 + .00033$

or $1.08228 \leq S \leq 1.08236$

(c) Using (3) It suffices to get error $R_n \leq \int_n^{\infty} \frac{1}{x^4}\,dx \leq .00001$

this is true if $\frac{1}{3n^3} < 10^{-5}$

$$\iff n > \sqrt[3]{\frac{10^5}{3}} \doteq 32.2$$

Since n must be a whole number, take $\underline{n \geq 33}$.