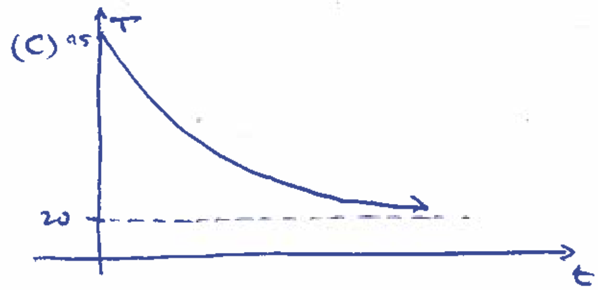
MATH 136, section 1 Problem Set 8, 'B' Solutions ①

7.1/14

(a) Newton's Law says the rate of change of the coffee's temperature, $T(t)$, is proportional to the difference $T(t)-20$. So the coffee cools the fastest at $t=0$ when $T(0)=95$

(b) $\dfrac{dT}{dt} = k(T - \underset{20}{\overset{\shortparallel}{A}})$ $(k<0)$

(c)

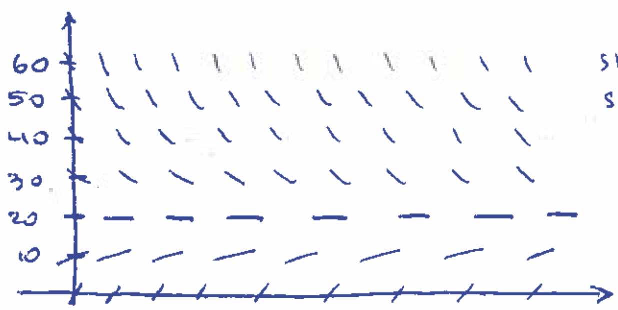


7.2/28 (a) From (b) above, if we know

$\dfrac{dT}{dt} = -1°/\text{minute}$ when $T = 70$,

$-1 = k(70-20)$, so $\boxed{k = -\dfrac{1}{50} = -.02}$

(b) the slope field looks roughly like this



$\dfrac{dT}{dt} = (-.02)(T-20)$

(c) Euler's method

$T_0 = 95.0$

$T(2) \doteq T_1 = 95 + (-.02)(95-20)(2) = 92.0$

$T(4) \doteq T_2 = 92.0 + (-.02)(92.0-20)(2) = 89.12$

$T(6) \doteq T_3 = 89.12 + (-.02)(89.12-20)(2) \doteq 86.36$

$T(8) \doteq T_4 = 86.36 + (-.02)(86.36-20)(2) \doteq 83.70$

$T(10) \doteq T_5 = 83.70 + (-.02)(83.70-20)(2) \doteq 81.15$

7.3/41

(a) when $a=b$, the equation is

$$\frac{dx}{dt} = k(a-x)^{3/2}$$

so $\int (a-x)^{-3/2}\,dx = \int k\,dt$

$$2(a-x)^{-1/2} = kt + C$$

when $t=0$, $x=0$, so

$$2/\sqrt{a} = C$$

$$\therefore \quad 2(a-x)^{-1/2} = kt + \frac{2}{\sqrt{a}}$$

$$(a-x)^{-1/2} = kt + \frac{1}{\sqrt{a}}$$

$$a-x = \frac{1}{\left(kt+\frac{1}{\sqrt{a}}\right)^2}$$

so $\boxed{x = a - \frac{1}{\left(kt+\frac{1}{\sqrt{a}}\right)^2}}$

(b) If $a>b$, then

$$\frac{dx}{dt} = k(a-x)(b-x)^{1/2}$$

so $\int \frac{dx}{(a-x)(b-x)^{1/2}} = \int k\,dt$

Let $u=(b-x)^{1/2}$, so $x=b-u^2$ and $a-x=(a-b)+u^2$

$dx = -2u\,du$

$$\int \frac{-2\cancel{u}\,du}{((a-b)+u^2)\cancel{u}} = \int k\,dt$$

this is the form given in #17 in the table since $a-b>0$, so $a-b=\left(\sqrt{a-b}\right)^2$.

Hence

$$\frac{-2}{\sqrt{a-b}} \tan^{-1}\left(\frac{\sqrt{b-x}}{\sqrt{a-b}}\right) = kt + C$$

From $x(0)=0$,

$$\frac{-2}{\sqrt{a-b}} \tan^{-1}\left(\frac{\sqrt{b}}{\sqrt{a-b}}\right) = C$$

so $$\boxed{t = \frac{2}{k\sqrt{a-b}}\left[\tan^{-1}\left(\frac{\sqrt{b}}{\sqrt{a-b}}\right) - \tan^{-1}\left(\frac{\sqrt{b-x}}{\sqrt{a-b}}\right)\right]}$$

42. We have $\frac{d^2T}{dr^2} + \frac{2}{r}\frac{dT}{dr} = 0$. Letting $S = \frac{dT}{dr}$, $\frac{dS}{dt} = \frac{d^2T}{dr^2}$, so this becomes the separable equation

$$\frac{dS}{dr} = -\frac{2}{r} S$$

$$\therefore \int \frac{dS}{S} = \int -\frac{2}{r}\, dr$$

$$\ln|S| = -2\ln r + C$$

so $S = \frac{k}{r^2}$ where $k = \pm e^C$

then $S = \frac{dT}{dr}$, so $\frac{dT}{dr} = \frac{k}{r^2}$. Now

we can simply integrate to solve for T:

$T = \frac{-k}{r} + l$. Finally, we are given

$T(1) = 15$ and $T(2) = 25$, so

$$\begin{cases} 15 = -k + l \\ 25 = -\frac{1}{2}k + l \end{cases}$$ subtract: $10 = k/2$ so $\boxed{\begin{array}{l} k = 20 \\ l = 35 \end{array}}$

$$\boxed{T(r) = \frac{-20}{r} + 35}$$