Math 136, Problem Set 5 'B' Solutions

§5.9/33 (a) With the given information, the largest n we can use for the midpoint rule is $n=4$, and then $\Delta x = 3.2/4 = .8$

$$\int_0^{3.2} f(x)\,dx \doteq (.8)\big(f(.4) + f(1.2) + f(2.0) + f(2.8)\big)$$

$$= (.8)(6.5 + 6.4 + 7.6 + 8.8)$$

$$= \boxed{23.44}$$

(b) Since $-4 \le f''(x) \le 1$, $|f''(x)| \le 4$ on $[0, 3.2]$. then

$$|E_M| \le \frac{4\cdot(3.2-0)^3}{24\cdot 4^2} \doteq \boxed{.3413} \quad (\text{using } K=4)$$

41. With n subdivisions, let $m_i = \frac{x_{i-1}+x_i}{2}$. then we have

$$\tfrac{1}{2}(T_n + M_n) = \tfrac{1}{2}\cdot \Delta x\big(f(x_0) + 2f(x_1) + \cdots + 2f(x_{n-1}) + f(x_n)\big) + \tfrac{1}{2}\Delta x\big(2f(m_1) + \cdots + 2f(m_n)\big)$$

$$= \frac{b-a}{2n}\big(f(x_0) + 2f(m_1) + 2f(x_1) + \cdots + 2f(x_{n-1}) + 2f(m_n) + f(x_n)\big)$$

this equals T_{2n} since $\frac{b-a}{2n}$ is the new Δx for a partition with $2n$ subintervals and m_i are the end points of the subdivided intervals.

§5.10/58 (Simpler but more "clever" method than the one I suggested in class) We must compute

$$\int_s^R (R-r)^2 \frac{r\,dr}{\sqrt{r^2-s^2}} = \lim_{a\to s^+} \int_a^R (R-r)^2 \cdot \frac{r\,dr}{\sqrt{r^2-s^2}}$$

Integrate by parts, where $u = (R-r)^2 \Rightarrow du = -2(R-r)\,dr$

$$dv = \frac{r\,dr}{\sqrt{r^2-s^2}} \Rightarrow v = \sqrt{r^2-s^2}$$

$$= \lim_{a\to s^+}\left[(R-r)^2\sqrt{r^2-s^2}\,\Big|_a^R + 2\int_a^R (R-r)\sqrt{r^2-s^2}\,dr \right]$$

$$= \lim_{a\to s^+}\left[0 - \overset{\to 0}{(R-a)\sqrt{a^2-s^2}} + 2R\int_a^R \sqrt{r^2-s^2}\,dr - 2\int_a^R r\sqrt{r^2-s^2}\,dr \right]$$

$$= \lim_{a\to s^+}\left[2R\int_a^R \sqrt{r^2-s^2}\,dr - 2\int_a^R r\sqrt{r^2-s^2}\,dr \right]$$

using #39 in the table and $u = r^2-s^2$ $du = 2r\,dr$

$$= \lim_{a\to s^+}\left[2R\left(\frac{r}{2}\sqrt{r^2-s^2} - \frac{s^2}{2}\ln\left|r+\sqrt{r^2-s^2}\right|\right)\Big|_a^R - \frac{2}{3}(r^2-s^2)^{3/2}\Big|_a^R \right]$$

$$= \lim_{a\to s^+}\Big[R^2\sqrt{R^2-s^2} - s^2R\ln\left|R+\sqrt{R^2-s^2}\right| - R^2\sqrt{a^2-s^2} + s^2R\ln\left|a+\sqrt{a^2-s^2}\right| - \frac{2}{3}(R^2-s^2)^{3/2} + \frac{2}{3}(a^2-s^2)^{3/2} \Big]$$

$$= R^2\sqrt{R^2-s^2} - s^2R\ln\left|R+\sqrt{R^2-s^2}\right| - 0 + s^2R\ln|s| - \frac{2}{3}(R^2-s^2)^{3/2} + 0$$

$$= \boxed{R^2\sqrt{R^2-s^2} - s^2R\ln\left|\frac{R+\sqrt{R^2-s^2}}{s}\right| - \frac{2}{3}(R^2-s^2)^{3/2}}$$

the first and 3rd terms can also be combined to

$$\left[R^2 - \frac{2}{3}(R^2-s^2)\right]\sqrt{R^2-s^2} = \frac{1}{3}(R^2+2s^2)\sqrt{R^2-s^2}$$

So another correct form is

$$\frac{1}{3}(R^2+2s^2)\sqrt{R^2-s^2} - s^2R\ln\left|\frac{R+\sqrt{R^2-s^2}}{s}\right|$$

61. Apply integration by parts with $u = x$, $dv = xe^{-x^2}\,dx$

then $du = dx$ and $v = -\frac{1}{2}e^{-x^2}$, so

$$\int_0^\infty x^2 e^{-x^2}\,dx = \lim_{b\to\infty} -\tfrac{1}{2}xe^{-x^2}\Big|_0^b + \tfrac{1}{2}\int_0^\infty e^{-x^2}\,dx$$

$$= \lim_{b\to\infty} -\tfrac{1}{2}\frac{b}{e^{b^2}} + \tfrac{1}{2}\int_0^\infty e^{-x^2}\,dx$$

By L'Hopital's Rule, this limit $\left(\frac{\infty}{\infty}\right)$ equals

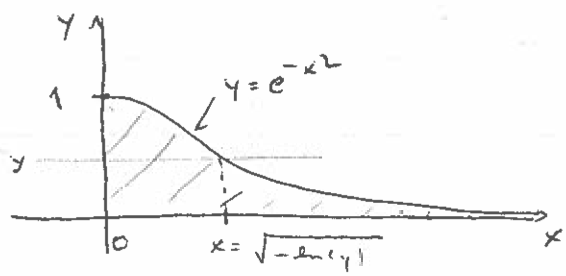
$$= \lim_{b\to\infty} -\tfrac{1}{2}\frac{1}{2be^{b^2}} + \tfrac{1}{2}\int_0^\infty e^{-x^2}\,dx$$

$$= 0 + \tfrac{1}{2}\int_0^\infty e^{-x^2}\,dx$$

$$= \tfrac{1}{2}\int_0^\infty e^{x^2}\,dx,$$ as was to be shown.

62. $\int_0^\infty e^{-x^2}\,dx$ computes the area bounded

by $y = e^{-x^2}$, $y = 0$, $x = 0$ and $x = \infty$:



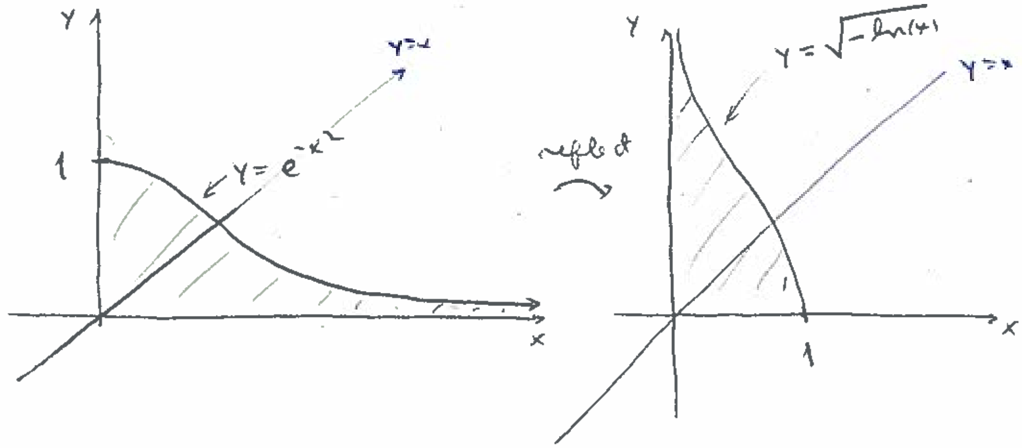
this part of the graph passes the horizontal line test, so we can solve for x in terms of y

$\ln(y) = -x^2$ $\quad$ ($0 \le y \le 1 \Rightarrow \ln(y) \le 0$)

so $\boxed{\sqrt{-\ln(y)} = x}$

$$\therefore \int_0^\infty e^{-x^2}\,dx = \int_0^1 \sqrt{-\ln(y)}\,dy.$$

(Another way to say this: e^{-x^2} has inverse function $\sqrt{-\ln(x)}$ on the domain $0 < x \le 1$. The graph is the region above, reflected across $y = x$:



Reflection doesn't change the area, so

$$\int_0^\infty e^{-x^2}\,dx = \int_0^1 \sqrt{-\ln(x)}\,dy$$