MATH 136, section 1 — PS 4 'B' Solutions ①

§5.7 /

12. $\displaystyle\int \frac{\sqrt{x^2-1}}{x^4}\,dx$ Let $x = \sec\theta$ $\quad dx = \sec\theta\tan\theta\,d\theta$

($\sqrt{u^2-a^2}$ form)

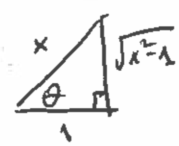
$$= \int \frac{\sqrt{\sec^2\theta - 1}}{\sec^4\theta}\cdot \sec\theta\tan\theta\,d\theta$$

$$= \int \frac{\tan^2\theta}{\sec^3\theta}\,d\theta$$

$$= \int \sin^2\theta\cos\theta\,d\theta \qquad \left(\text{since } \tan\theta = \frac{\sin\theta}{\cos\theta}\right.$$

this is $\int u^2\,du$ for $u = \sin\theta$ $\qquad \left.\sec\theta = \frac{1}{\cos\theta}\right)$

$$= \frac{1}{3}\sin^3\theta + C$$



$\frac{x}{1} = \sec\theta = \frac{\text{hyp}}{\text{adj}}$

$$= \frac{1}{3}\left(\frac{\sqrt{x^2-1}}{x}\right)^3 + C$$

$$= \boxed{\frac{(x^2-1)^{3/2}}{3x^3} + C}$$

13. $\displaystyle\int \frac{1}{x^2\sqrt{x^2+4}}\,dx$ $\quad \sqrt{u^2+a^2}$ form, so let $x = 2\tan\theta$

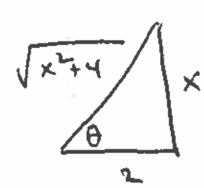
$dx = 2\sec^2\theta\,d\theta$

$$= \int \frac{1}{4\tan^2\theta\sqrt{4\tan^2\theta+4}}\cdot 2\sec^2\theta\,d\theta$$

$$= \frac{1}{4}\int \frac{\sec\theta}{\tan^2\theta}\,d\theta$$

$$= \frac{1}{4}\int \frac{\cos\theta}{\sin^2\theta}\,d\theta \longleftarrow \text{this is } \int u^{-2}\,du \text{ for } u = \sin\theta$$

$$= \frac{-1}{4}\cdot\frac{1}{\sin\theta} + C$$



$\frac{x}{2} = \tan\theta$

$$= \boxed{-\frac{\sqrt{x^2+4}}{4x} + C}$$

16. $$\int_0^{2\sqrt{3}} \frac{x^3}{\sqrt{16-x^2}}\,dx$$

$\sqrt{a^2-u^2}$ form, so let

$x = 4\sin\theta \quad dx = 4\cos\theta\,d\theta$

$x=0 \leftrightarrow \theta = 0$

$x = 2\sqrt{3} \leftrightarrow \sin\theta = \frac{\sqrt{3}}{2}$ so $\theta = \frac{\pi}{3}$

$$= \int_0^{\pi/3} \frac{64\sin^3\theta}{\sqrt{4-4\sin^2\theta}} \cdot 2\cos\theta\,d\theta$$

$$= 64\int_0^{\pi/3} \sin^3\theta\,d\theta$$

$$= 64\int_0^{\pi/3} (1-\cos^2\theta)\cdot\sin\theta\,d\theta$$

$$= 64\left[\int_0^{\pi/3} \sin\theta\,d\theta - \int_0^{\pi/3} \cos^2\theta\sin\theta\,d\theta\right]$$

$$= 64\left[-\cos\theta\Big|_0^{\pi/3} + \frac{1}{3}\cos^3\theta\Big|_0^{\pi/3}\right]$$

$$= 64\left[-\frac{1}{2}+1 + \frac{1}{24} - \frac{1}{3}\right]$$

$$= 64\cdot\frac{5}{24} = \boxed{\frac{40}{3}}$$

35. Completing the square, $x^2+x+1 = (x+\frac{1}{2})^2 + \frac{3}{4}$

If you do this one "from scratch," not using the table, let $x+\frac{1}{2} = \frac{\sqrt{3}}{2}\tan\theta$, then $dx = \frac{\sqrt{3}}{2}\sec^2\theta\,d\theta$

$$\int \frac{dx}{x^2+x+1} = \int \frac{dx}{(x+\frac{1}{2})^2+\frac{3}{4}}$$

$$= \int \frac{\frac{\sqrt{3}}{2}\sec^2\theta\,d\theta}{\frac{3}{4}(\tan^2\theta+1)}$$

since $\tan^2\theta + 1 = \sec^2\theta$

$$= \frac{2}{\sqrt{3}}\int d\theta$$

$$= \frac{2}{\sqrt{3}}\theta + C = \boxed{\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right) + C}$$

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36. $$\int \frac{x}{\sqrt{3-2x-x^2}}\,dx$$

Completing the square, $3-2x-x^2 = -(x^2+2x-3)$
$= -((x+1)^2-4)$
$= 4-(x+1)^2$

$$= \int \frac{x}{\sqrt{4-(x+1)^2}}\,dx$$

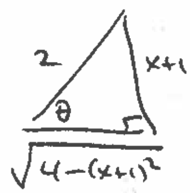
$\sqrt{a^2-u^2}$ form, so let $x+1 = 2\sin\theta$

$\Rightarrow \begin{cases} dx = 2\cos\theta\, d\theta \\ x = 2\sin\theta - 1 \end{cases}$

$$= \int \frac{(2\sin\theta-1)\cdot 2\cos\theta\, d\theta}{\sqrt{4-4\sin^2\theta}}$$

$$= \int (2\sin\theta - 1)\,d\theta$$

$$= -2\cos\theta - \theta + C$$



$\frac{x+1}{2} = \sin\theta$

$\sin^{-1}\left(\frac{x+1}{2}\right) = \theta$

$$= -\sqrt{4-(x+1)^2} - \sin^{-1}\left(\frac{x+1}{2}\right) + C$$

$$= \boxed{-\sqrt{3-2x-x^2} - \sin^{-1}\left(\frac{x+1}{2}\right) + C}$$