MATH 136 Problem Set 3 'B' Solutions

5.5/71. We want to show that if $a, b \geq 0$, then

$$\int_0^1 x^a (1-x)^b \, dx = \int_0^1 x^b (1-x)^a \, dx$$

In the left-hand integral, let $u = 1-x$, so $x = 1-u$ and $dx = -du$. Then

$$\int_0^1 x^a (1-x)^b \, dx = \int_{u=1}^{u=0} (1-u)^a u^b \cdot (-du)$$

$$= \int_0^1 u^b (1-u)^a \, du$$ (interchanging limits cancels sign on du)

Since this is a definite integral, it equals $\int_0^1 x^b (1-x)^a \, dx$, as we wanted to show.

5.5/72 (a) If we reflect the graph $y = \sin x$ across the vertical line $x = \frac{\pi}{4}$, we get the cosine graph $y = \cos x$. that is: $\sin(\frac{\pi}{2} - x) = \cos x$. So

$$\int_0^{\pi/2} f(\cos x) \, dx = \int_0^{\pi/2} f(\sin(\tfrac{\pi}{2} - x)) \, dx$$

Now let $u = \frac{\pi}{2} - x$, so $du = -dx$; the right-hand integral becomes $= \int_{\pi/2}^{0} f(\sin(u))(-du)$. As in 5.5/71, reversing the order of the limits cancels the negative sign with du, so

$$\int_0^{\pi/2} f(\cos x) \, dx = \int_0^{\pi/2} f(\sin u) \, du$$

$$= \int_0^{\pi/2} f(\sin x) \, dx.$$

(b) By part (a) with $f(x) = x^2$, $\int_0^{\pi/2} \sin^2 x \, dx = \int_0^{\pi/2} \cos^2 x \, dx$

But $\int_0^{\pi/2} \cos^2 x \, dx + \int_0^{\pi/2} \sin^2 x \, dx = \int_0^{\pi/2} 1 \, dx = \pi/2$. So both integrals $= \boxed{\frac{\pi}{4}}$.

5.6/40. For $\int x^n e^x\,dx$, let $u = x^n$, $dv = e^x\,dx$

then $du = n x^{n-1}$, $v = e^x$, so by the parts

formula $\int x^n e^x\,dx = x^n e^x - \int n x^{n-1} e^x\,dx$

$$= x^n e^x - n \int x^{n-1} e^x\,dx$$

this is the <u>reduction formula</u> for a power of x

times an exponential.

5.6/44. (Comment: the key to this one is not

letting the "noisy" numbers overwhelm you).

We have $v(t) = -gt - v_e \ln\left(\frac{m - rt}{m}\right)$

$$= -gt - v_e \ln\left(1 - \frac{rt}{m}\right).$$

We want first to integrate $\int v(t)\,dt$.

$$S(t) = \int v(t)\,dt = \int -gt - v_e \ln\left(1 - \frac{rt}{m}\right) dt$$

$$= -\frac{gt^2}{2} - v_e \int \ln\left(1 - \frac{rt}{m}\right) dt$$ if we let $w = 1 - \frac{rt}{m}$

$dw = -\frac{r}{m}\,dt$

$$= -\frac{gt^2}{2} + \frac{m v_e}{r} \int \ln w\,dw$$ $-\frac{m}{r}\,dw = dt$

using parts now, $u = \ln w$ $\quad dv = dw$

$du = \frac{1}{w}\,dw$ $\quad v = w$

$$= -\frac{gt^2}{2} + \frac{m v_e}{r}\left[w \ln w - \int w \cdot \frac{1}{w}\,dw\right]$$

$$= -\frac{gt^2}{2} + \frac{m v_e}{r}\left[w \ln w - \int dw\right]$$

$$= -\frac{gt^2}{2} + \frac{m v_e}{r}\left[w \ln w - w\right]$$

$$S(t) = -\frac{gt^2}{2} + \frac{m v_e}{r}\left[\left(1 - \frac{rt}{m}\right)\ln\left(1 - \frac{rt}{m}\right) - \left(1 - \frac{rt}{m}\right)\right]$$

We want $S(60) - S(0) \doteq \frac{-(9.8)(60)^2}{2} + \frac{(3\times10^4)(3\times10^3)}{160}\left[\left(1 - \frac{(160)(60)}{3\times10^4}\right)\ln(") - (")\right]$

$$- \left[0 + \frac{(3\times10^4)(3\times10^3)}{160}\left(1 \ln(1) - 1\right)\right]$$

$\doteq 14{,}844$ m.

5.6/47. Assuming $f(0) = g(0) = 0$

$\int_0^a f(x)\, g''(x)\, dx$ can be done by parts, where
$u = f(x)$, $dv = g''(x)$
$du = f'(x)\,dx$ $v = g'(x)$

$= f(x)\, g'(x)\Big|_0^a - \int_0^a f'(x)\, g'(x)\, dx$

$= f(a)\, g'(a) - \cancel{f(0)}\overset{0}{}\, g'(0) - \int_0^a f'(x) g'(x)\, dx$

use parts again with
$u = f'(x)$ $dv = g'(x)\,dx$
$du = f''(x)\,dx$ $v = g(x)$

$= f(a)\, g'(a) - f'(x)\, g(x)\Big|_0^a + \int_0^a f''(x)\, g(x)\, dx$

$= f(a)\, g'(a) - f'(a)\, g(a) + \int_0^a f''(x)\, g(x)\, dx,$
$+ f'(0)\cancel{g(0)}\overset{0}{}$

which is what we wanted to show.