MATH 136, section 1, Problem Set 7 'B' Solutions 1

§6.4 / 34b

We have $x(t) = \int_0^t \cos(\pi u^2/2)\,du$

$y(t) = \int_0^t \sin(\pi u^2/2)\,du$

So by the FTC, part 1,

$$\frac{dx}{dt} = \cos\left(\frac{\pi t^2}{2}\right), \quad \frac{dy}{dt} = \sin\left(\frac{\pi t^2}{2}\right)$$

The arclength is

$$L = \int_0^t \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$$

$$= \int_0^t \sqrt{\cos^2(\pi t^2/2) + \sin^2(\pi t^2/2)}\,dt$$

$$= \int_0^t 1\,dt \quad \text{(trig identity :))}$$

$$= \boxed{t}.$$

§6.5 / 20

$V(r) = \frac{P}{4\eta l}(R^2 - r^2)$, so the average

velocity is

$$V_{ave} = \frac{1}{R}\int_0^R \frac{P}{4\eta l}(R^2 - r^2)\,dr$$

$$= \frac{P}{4\eta l R}\left[R^2 r - \frac{r^3}{3}\right]\Big|_0^R$$

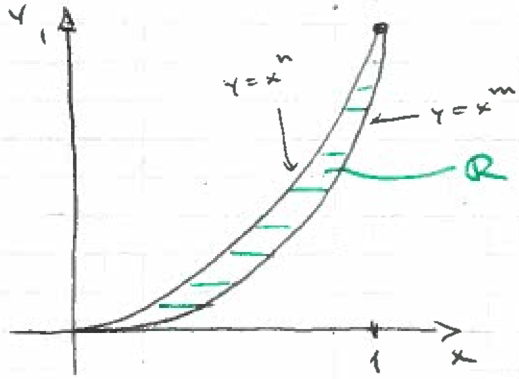
$$= \frac{2PR^3}{12\eta l R} = \boxed{\frac{PR^2}{6\eta l}}$$

The maximum velocity occurs when $r=0$, since $V(r) = \frac{P}{4\eta\ell}(R^2-r^2) < V(0)$ if $r>0$. So

$\boxed{V_{max} = V(0) = \frac{PR^2}{4\eta\ell}.}$ This shows $v_{ave} = \frac{PR^2}{6\eta\ell}$

$= \frac{2}{3} V_{max}$.

§6.6/52 (a) Since $m > n \geq 0$ are integers, the region R looks like this:



(Since $0 \leq x \leq 1$, the larger power gives a smaller y, so $y=x^n$ lies above $y=x^m$.)

(b) Using the formulas given in #51,

$$A = \int_0^1 x^n - x^m \, dx = \frac{x^{n+1}}{n+1} - \frac{x^{m+1}}{m+1}\Big|_0^1 = \frac{1}{n+1} - \frac{1}{m+1}$$

$$= \boxed{\frac{m-n}{(m+1)(n+1)}}.$$

$$\overline{X} = \frac{(m+1)(n+1)}{m-n} \int_0^1 x(x^n - x^m)\, dx = \frac{(m+1)(n+1)}{m-n} \cdot \left(\frac{x^{n+2}}{n+2} - \frac{x^{m+2}}{m+2}\right)\Big|_0^1$$

$$= \frac{(m+1)(n+1)}{m-n} \cdot \frac{(m-n)}{(m+2)(n+2)} = \boxed{\frac{(m+1)(n+1)}{(m+2)(n+2)}}$$

$$\overline{Y} = \frac{(m+1)(n+1)}{m-n} \int_0^1 \frac{1}{2}(x^{2n} - x^{2m})\, dx$$

$$= \frac{(m+1)(n+1)}{2(m-n)} \left(\frac{x^{2n+1}}{2n+1} - \frac{x^{2m+1}}{2m+1}\right)\Big|_0^1$$

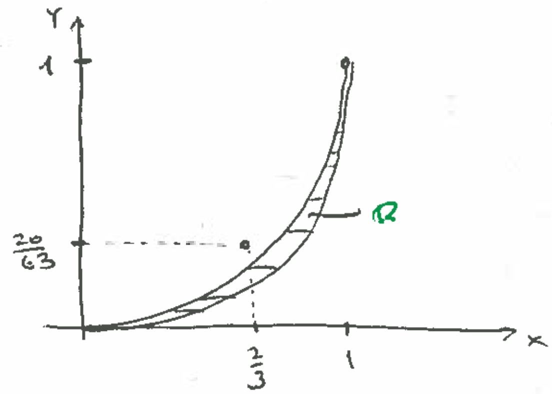
$$= \boxed{\frac{(m+1)(n+1)}{(2m+1)(2n+1)}}.$$

(c) If $m = 4$, $n = 3$, for instance, we get

$$\bar{X} = \frac{5\cdot 4}{6\cdot 5} = \frac{2}{3} \qquad \bar{Y} = \frac{5\cdot 4}{9\cdot 7} = \frac{20}{63}$$

$\bar{x}^3 = \frac{8}{27}$, $\bar{x}^4 = \frac{16}{81}$; But clearly,

$\underset{.317}{\bar{y}}$ does not lie between $\underset{.296}{\bar{x}^3}$ and $\underset{.198}{\bar{x}^4}$. therefore the point $(\bar{x}, \bar{y})$ lies above $y = x^3$. (This is possible because of the curved shape of R:



)

§6.8 / 8

a) $f(x) \geq 0$ for all x by definition and

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_0^{10} f(x)\,dx = \tfrac{1}{2}(6)(.2) + \tfrac{1}{2}(4)(.2) = 1$$

So $f(x)$ is a pdf.

(b) $P(X < 3) = \int_0^3 f(x)\,dx = \frac{1}{2}(3)(.1) = \frac{3}{20} = \boxed{.15}$

$$P(3 \leq X \leq 8) = \int_3^8 f(x)\,dx = \left(\frac{.1+.2}{2}\right)(3) + \left(\frac{.1+.2}{2}\right)(2)$$

$$= \frac{3}{4} = \boxed{.75}$$

c) $$\bar{x} = \int_{-\infty}^{\infty} x f(x)\,dx = \int_0^6 x\cdot\frac{x}{30}\,dx + \int_6^{10} x\cdot\left(-\frac{1}{20}(x-10)\right)dx$$

$$= \frac{x^3}{90}\Big|_0^6 + \left(\frac{x^2}{4} - \frac{x^3}{60}\right)\Big|_6^{10}$$

$$= \frac{12}{5} + 25 - \frac{50}{3} - 9 + \frac{18}{5} = \frac{36+375-250-135+54}{15}$$

$$= \frac{80}{15} = \boxed{\frac{16}{3}}$$

(N.B. $\bar{x}$ must be in $[0, 10]$)